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March 1977

STUDIES IN ADVANCED PHOTOGRAMMETRIC TRIANGULATION TECHNIQUES

DBA Systems Inc.

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Analytical photogrammetry, spurred on by a rapidly evolving computer technology has achieved major advances within the past decade. These advances have been well documented both in Europe as well as the United States. Contained in this report however, is an attempt to define those near term challenges brought about by larger data sets, such developments as the Global Positioning System, and continued improvement in auxiliary sensors (e.g., inertial platforms, stato- next page

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scope units, etc.). Continued research commensurate with these challenges are of paramount importance where a first order, wide area mapping requirement exists.

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EVALUATION

For the past few years, RADC has concentrated its activities on critical problem areas within the field of analytical photogrammetry. In order to maintain the technology base which has successfully provided solutions to the above, a continual reexamination of the state-of-the-art is essential. Under a modest level of effort, DBA Systems Inc. has developed a snapshot of current capabilities and has provided a roadmap of developmental activity. Related technology of interest to the mapping community such as the Global positioning system, improved inertial navigation systems and others have also been drawn upon for improving current triangulation procedures. Recommendations offered by the contractor are being seriously entertained for pursuit under RADC's exploratory research program and in consonance with TPO 2.

John R Callander

JOHN R. CALLANDER
Project Engineer

1.0 INTRODUCTION

The objective of this study is to identify new techniques in analytical photogrammetry which offer significant improvements in target location accuracy and/or drastically reduce data reduction time for large data cases. In order to satisfy these broad objectives, DBA has reviewed the current analytical triangulation procedures in use by the USAF, DMA and other Governmental and Commercial organizations. The results of this review are presented in Section 2 in the form of a basic analytical triangulation model directed at the use of Aerial Photography.

Once the current procedures were broadly defined, DBA proceeded to analyze each step to determine the major shortfalls and attempts to constructively identify alternative techniques to alleviate these shortfalls. The results of this analysis are presented in Section 3.

Sections 2 and 3 provide background and the basic organizational framework for Section 4 which discusses in detail the alternative techniques. Some of the techniques described in Section 4 are very detailed, whereas others are only conceptual and would require more in-depth study before their ultimate utility can be evaluated.

The conclusions and recommendations resulting from this study are presented in Section 5.

The primary shortfall which is identified in Section 3 relates to the inability of most aerotriangulation programs to adequately account for systematic errors in the observations. Specific examples of these errors are film deformation, camera calibration parameters, and most importantly, systematic errors in auxiliary observations such as GPS positioning, inertial survey data and precise attitude sensor data etc. There are a number of alternative techniques available for modeling these systematic errors. These alternative techniques are discussed in Section 4 of this report. Some of these techniques create computational problems in that new algorithms are required to efficiently solve the least squares problem. Of particular interest is an efficient algorithm to solve banded-bordered normal equations which have large highly patterned borders.

Other shortfalls which are considered, relate primarily to increasing the overall throughput of the entire point positioning process, not simply reducing the data reduction time. Included in this area are techniques for identifying available coverage of an area, automatic and semi-automatic editing schemes, and automatic data reordering algorithms.

Another broad area which is considered in this report is the incorporation of object space constraints in a general photogrammetric reduction. This discussion is not limited to the normal distance, azimuth and equal elevation constraints (although they are discussed), but includes a general technique for including geometric type constraints with specific reference to digital data collection schemes.

Naturally all items discussed do not receive equally detailed development, however, sufficient information is presented such that the general approach can be understood for all areas. In some cases detailed mathematical developments are presented, whereas, in others only a conceptual analysis is presented. Those areas discussed in conceptual terms fall into two broad categories. They are either standard techniques which have been documented elsewhere and not implemented in an aerotriangulation reduction, or, the concept is very new and there were insufficient resources to develop the details completely for presentation at this time.

2.0 CURRENT ANALYTICAL AEROTRIANGULATION

For the purpose of this report, we will define those techniques for precisely determining the position of targets identified on aerial photographs as an Analytical Aerotriangulation System. Figure 2.0-1 diagrammatically illustrates this system and identifies ten major processes of the system. These ten processes can further be broken into a number of steps, each of which can be accomplished in a variety of ways. The techniques employed during any one step often have a direct influence on the techniques to be employed during other steps or subsequent processes.

The interelationship between the processes, steps and techniques of this system operates in both directions. It is obvious that alternatives selected at each step/process can define or limit the alternatives available for subsequent steps/processes. Experience has dictated that, although not as obvious and frequently overlooked, the alternatives selected for subsequent processes may also influence selection of alternatives for prior processes. That is to say, a prior knowledge of the constraints on a final product may define the methods by which the product is attained. This is particularly true for economical and timely satisfaction of requirements in target location.

As a basic example of this reverse relationship, let us assume that accuracy specifications for point locations have been defined. In order to meet these specifications one must consider the block adjustment technique to be employed as well as the scale of the photography required to accommodate

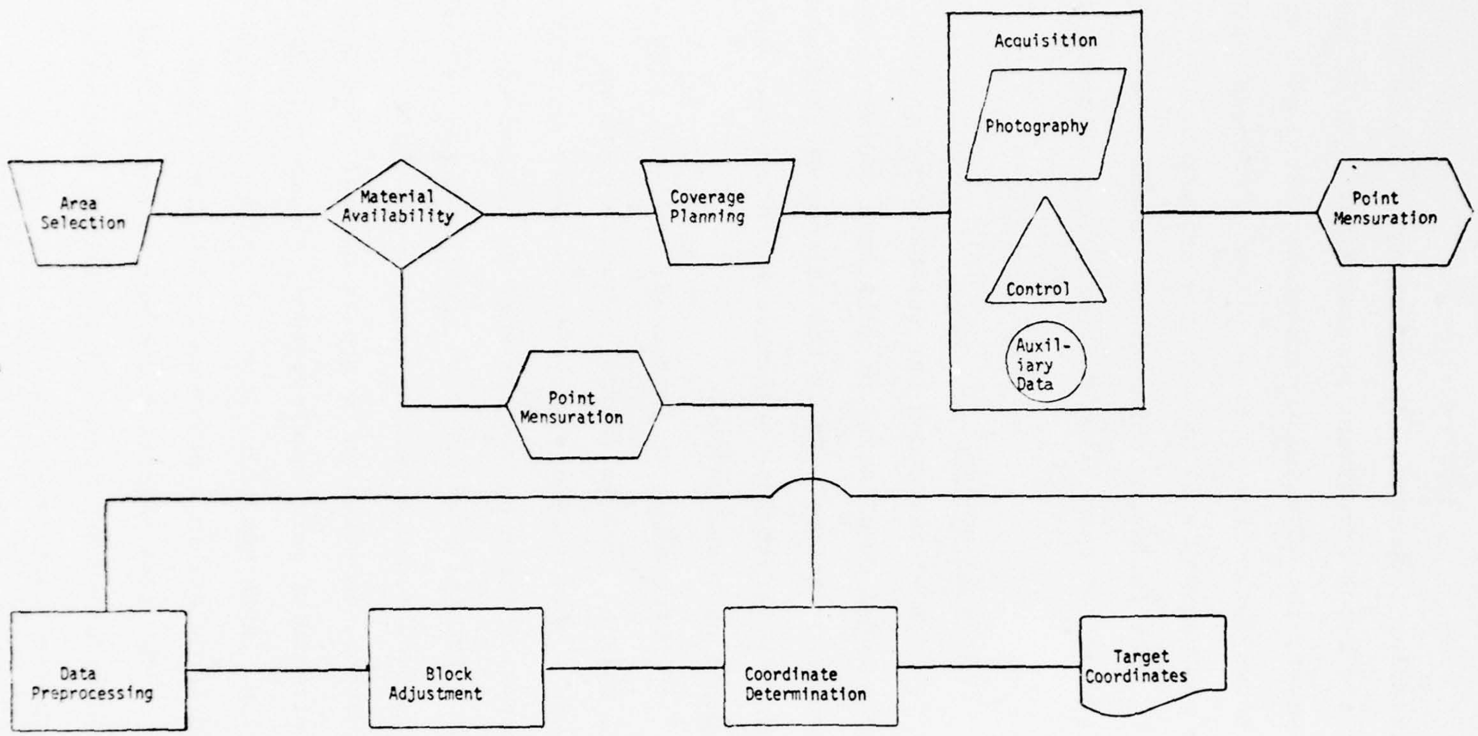

FIGURE 2.0-1
PHOTOGRAMMETRIC TARGET LOCATION PROCESS

that particular technique. In referring to Figure 2.0-1, we see that the block adjustment process falls late in the system flow, however, the adjustment parameters, along with scale, must be known during the coverage planning process such that adequate photography will support the adjustment.

The following subsections present brief descriptions of each process of comprising the Analytical Aerotriangulation System.

2.1 Area Selection

In most instances the selection of an area for target location is based on military or political decisions and not on photogrammetric factors. It is the science of photogrammetry which supports target area selection and not target area selection that supports photogrammetry.

Photogrammetric factors do, however, play a very important role in area selection if one is planning to calibrate or evaluate a photogrammetric system or technique. In this circumstance, certain parameters must be defined and conditions satisfied. For calibration, such parameters may consist of adequate ground control and the geometry of the ground control. For evaluation, in addition to ground control, one must consider various altitudes, terrain elevations, slopes and auxiliary data acquisition. Such considerations, in addition to others, are discussed in detail in a previous report (DBA, 1974).

The Area Selection process is not analyzed in this report and is presented merely as a logical introduction to the total Aerotriangulation System.

2.2 Material Availability

Upon selection of an area of interest, one must determine what support material is available and must analyze the suitability of any such material. Currently, material availability is determined through incomplete files of past flights and the experience and knowledge of personnel assigned to make this determination. Analysis is based on scale, coverage, clarity (resolution), amount of cloud cover, completeness of coverage ("Holiday" areas), etc. The majority of this process is dependent on human judgement and reliability.

2.3 Coverage Planning

Should the material availability process prove that coverage is inadequate, the process of coverage planning is initiated. The following factors must be considered during this step:

1. Scale of photography
2. Ground resolution
3. Photo Sensor type and availability
4. Control availability
5. Auxiliary data collection system type and availability
6. Climatic considerations
7. Time considerations

Accuracy requirements for target location frequently dictate the scale of the photography, and to some extent the resolution specifications. The planning of missions must include an analysis of what types of sensors are available for acquiring the data. Included in the sensor availability must be considerations for the logistics of using them. The sensor availability process is often limited to a single sensor, as it may be the only sensor which can collect the required data over the specific area. This default decision then leads to the development of techniques for exploitation of a specific sensor.

Also included in this evaluation must be a consideration for the availability of a data reduction program which can accommodate this sensor and obtain the required accuracy. Additional thought must be given to the calibration of the sensor and/or its calibratability during the coverage planning process.

Another area which must be considered during this planning process is the availability of auxiliary exposure station data. As with sensor availability, consideration must be given to the ability to exploit this data once it has been gathered.

The available ground control must also be considered in planning the acquisition of photographic coverage. This consideration must include the type, accuracy and distribution of such control.

Climatic considerations are of primary concern in planning photographic missions. Although, there is no control over this, the mission can be scheduled for the most promising time periods and to accommodate reasonable contingencies.

Although listed last, probably the most important factor in planning coverage missions is the time factor (i.e. when is the requirement to be satisfied). This can have a direct bearing on the sensor used, on the control used and other factors.

As can be seen from this brief review of the coverage planning process, the techniques available and the planners' understanding of them have a direct effect on the decisions made and an indirect influence on all subsequent processes.

2.4 Acquisition

Material acquisition pertains to the required photography as well as to the auxiliary data and ground control. For the purpose of this report, film processing, will be included. An initial review of the coverage is performed to determine utility of the film for the specified project. During this process, all decisions made during coverage planning are combined into a single system. This timely combination of equipment, personnel and techniques may be dictated by weather conditions.

2.5 Point Selection, Identification and Mensuration

Once the photographic imagery is available, the next step in the photogrammetric targeting problem is the selection, identification and mensuration of the control points, pass points and target points.

The ground control must be positively identified on the photograph. Typically this is a manual operation which requires the operator to stereoscopically view a pair of overlapping photographs and either measure the point stereoscopically or mark the imagery for subsequent measurement.

An alternative used occasionally is the signalization of the ground control. Some experiments have also been performed with signalized pass points (i.e. non-control points). In general, this is impractical due to the time requirements and inaccessibility of the area.

2.6 Data Preprocessing

Before a least squares adjustment can be initiated, the data must be suitably transformed, generated, edited and arranged so that efficient algorithms can be utilized for the least squares process. The specific techniques and models utilized in this phase must, of necessity, be matched to the adjustment algorithm. For this reason, computational routines for data preprocessing are frequently appended to the "front end" of the least squares adjustment program. However, for economic reasons , data preprocessing algorithms are just as often small programs which are executed independent of the adjustment. (It is often more easy to process a number of small computer runs through a central computer facility than a long run and it is easier to edit data and rerun a small program than a large one).

In order to fully understand the data preprocessing phase of an Aerotriangulation one must first understand the data, the mathematical model and the least squares algorithm implemented in the form of a block adjustment program.

There are four basic steps required in the preprocessing phase of an adjustment. These are:

1. Data transformation,
2. Generation of initial approximations,
3. Data editing, and
4. Data arrangement.

The data transformation step converts the observations and associated weight matrices to a reference system consistent with the mathematical model utilized by the block adjustment.

The generation of initial approximations for all parameters (unknowns) is required because in general the photogrammetric observation equations are non-linear with respect to the parameters and must be iteratively solved using a linearized set of equations. The basic equations are linearized about some arbitrary initial approximations. Corrections to the initial approximations are computed and applied to these arbitrary values and the solution respectively performed until the corrections become sufficiently small. Obviously, the closer the values of these arbitrary initial approximations are to the true value the fewer the number of iterations will be required. The problem as far as data preprocessing is concerned, then, is one of obtaining the "best" set of initial values as possible for all parameters. The obvious solution is to set the initial value equal to the observed value wherever possible, however, for some classes of parameters it is more difficult to come up with observations than it is to simply "guess" at initial values. Initial approximations for still other classes of parameters can be easily computed based upon the use of approximate mathematical models.

The detection and elimination of aberrant observations (data editing) is performed during this portion of aerotriangulation to minimize the run time (and rerun time) for the actual block adjustment program. This data editing is frequently hampered by the quality of the initial approximation and the fact that some initial approximations are actually derived from the observations.

The last step in data preprocessing is the arrangement and storage of all the data in a format and on a media suitable for efficient solution of the adjustment. To fully comprehend this step one must be intimately familiar not only with the mathematical model and the least squares process, but also the specific implementation algorithms included in the computational reduction. For discussion herein it is assumed that one is familiar with the least squares solution and the specific nature of normal equations and the manner in which they can be reduced to a banded structure (Gyer et.al, 1969). The problem of arranging the data becomes one of logically ordering the photographs in a block to minimize the size of the bandwith associated with this reduced set of normal equations.

2.6.1 Reordering

An analytical aerotriangulation is based on the solution of a system of linearized normal equations, the formation of which will be discussed later in this report. The parameters to be solved are usually iterative corrections or adjustments to orientation parameters of the individual photographs. In general, the coefficient matrix for this system is diagonally banded and symmetrical. The elements along the main diagonal are coefficients of the corrections to the orientation parameters and the off diagonal elements represent correlations between the parameters.

It has been found that the speed and efficiency with which these normal equations can be solved is directly related to the bandwidth of the coefficient matrix; the bandwidth being the furtherest distance of a correlation element from the main diagonal. When the orientation parameters are grouped by individual photo, as they usually are, the correlation elements furthest from the diagonal represent correlations between parameters of different photos. This correlation is a consequence of the photos having imaged a common object point. In order to minimize the bandwidth and maximize the speed of the solution, it is desirable to arrange the parameters in the matrix so that those for highly correlated photos (having common image points) are as close together as possible. The initial arrangement of the parameters is based on a pre-established ordering or numbering of the photos. If this does not provide the smallest possible bandwidth, some rearranging or reordering is desirable.

The initial ordering of the photos is normally established by a sequential photo numbering scheme introduced during mensuration. The scheme used is usually devised for convenience in mensuration and is not always the optimum for the solution. The most common scheme is down strip photo numbering in which the photos in each strip are numbered consecutively in the same direction and the numbering sequence is continued from one strip to the next.

Down strip numbering has proven to be optimum only for very regular rectangular blocks in which the number of photos per strip (p) and the number of strip(s) are related by $p > 2s-1$. For shorter broader blocks in which $p < 2s-1$, reordering into a cross strip numbering scheme is desirable. When more complex correlations are introduced through cross-strips or other irregularities, reordering by more intricate techniques is required.

Most commercial and Government block adjustment programs require an intuitive assessment of the best numbering scheme and manual ordering or reordering of the input data. Some automation and sophistication has been introduced in a few instances. SAPGO (Wong & Elphingstone, 1972) has incorporated an automatic cross-strip numbering capability. In order to accommodate some irregularities between strips, cross-strip numbering is accomplished by setting up a series of equally spaced parallel lines perpendicular to the flight line of the first strip. The parallel lines are spaced according to the average base length of the first strip and are used as numbering zones. All of the exposure stations that fall within a zone are numbered in a sequential manner. An even more universal technique has been incorporated into program COMBAT (Brown, 1974). This technique analyzes the off-diagonal correlation elements in the actual coefficient matrix. In minimizing the bandwidth, the longest rows in the matrix are moved to the center of the matrix and the shortest rows to the edge. The algorithm checks subsequent rows to see if they should be interchanged. An interchange will take place only if it will move a longer row closer to the center of the matrix. With a matrix of variable bandwidth the final reordered matrix would emerge appearing in a diamond form. Experience shows that this technique consistently produces the minimum bandwidth ordering scheme.

2.7 Block Adjustments

There are four basic categories of Analytical Aerotriangulation as represented in Table 2.7-1. The first class is intended to encompass all methods that entail piecewise (as opposed to simultaneous) processes of adjustment. It includes most methods for computational connection of independently established models (i.e., semi-analytical methods) as well as methods involving numerical relative orientation (i.e., 'cantilevering') of successive photos within each strip followed by reconciliation of discrepancies within and between strips by various polynomial transformations.

Methods of Class 2 involve the assembly of independently established models by means of the simultaneous, least squares determination of the particular sets of similarity transformations that best interconnect overlapping models while at the same time providing the best overall accommodation of the block as a whole to available control. (A similarity transformation performs translations , rotations and a change of scale.) A general account of such methods is given by Ackermann (1972). Although technically classifiable as semi-analytical methods (by virtue of the instrumentation formation of individiual models), they differ from the semi-analytical methods of Class 1 in a significant way, namely, in the simultaneity of the recovery of all interconnecting transformations. Because of this, the methods of Class 2 have proven to be decidedly more effective than those of Class 1 and accordingly merit a separate classification.

In the United States the expression "simultaneous

TABLE 2.7-1.

CLASSIFICATION OF METHODS OF ANALYTICAL AEROTRIANGULATION

CLASS	METHODS	EXAMPLES OF COMPUTER PROGRAMS*
1	Strip Assembly by Connective Polynomial Transformations	NRC System (National Research Council, Canada) USC&GS Three Photo Relative Orientation
2	Simultaneous Adjustment of Independent Models by Similarity Transformations	ANBLOCK (Vand den Haut) PAT-M Series (Ackermann, et. al., Stuttgart University)
3	Bundle Adjustment	BSOR-BLOCK (DBA/RADC/ACIC) USC&GS Block Aerotriangulation (NOS) MUSAT (ETL/TOPOCOM) SURBAT (DBA/RADC/ACIC) COMBAT (DBA)
4	Bundle Adjustment with Self-Calibration	STEREO (DBA) SMAC (DBA/ETL) COMBAT II (DBA) BAP (Bauer & Müller, Hannover,W.G.) Generalized MUSAT(Automatic) STEREO with Geometric Constraints(DBA)

* This listing is not intended to be all-inclusive.

block aerotrinagulation: is commonly used to designate the methods of Class 3; Europeans tend to favor the more concise expression "bundle adjustment" that is used in Table 1. Methods of Class 3 exercise the utmost degree of mathematical and statistical rigor. This involves the simultaneous least squares triangulation of all bundles of rays from all exposure stations to all measured points coupled with the simultaneous accomplishment of the resection of all participating photos and adjustment of the ground survey.

The methods of Class 3 are based on the common assumption that measured film coordinates have been adequately corrected for all significant sources of systematic error and thus are contaminated only by random error. Those of Class 4 recognize that this is not necessarily the case in practice; unresolved systematic errors in film coordinates often assume appreciable significance and must be accorded due consideration if optimum results are to be produced. This can be accomplished in considerable measure by a process wherein additional parameters consisting of coefficients of appropriate error models are recovered within the bundle adjustment itself. Such recovery constitutes a process of self-calibration and hence the designation "bundle adjustment with self-calibration". This last catagory of programs is of particular importance with regard to improving the accuracy of point positioning. Two programs of specific interest are the Generalized MUSAT (Greve et.al 1976) and the program STEREO with Geometric constraints (DBA, 1972). These programs contain provisions for constraining object space points to lie on specific geometric surfaces such as planes, circular cylinders, etc.

In addition to these general categories, special programs have been developed for lunar orbiter and Apollo Photography which incorporate orbital constraints to force all exposure stations to lie along an arc defined by six continuous orbital elements. These orbital parameters replace the three exposure station position parameters associated with each photograph and add six parameters per strip. These additional parameters are placed in the border of the normal equation (See Section 4.6 for a detailed development of the banded-bordered normal equations).

DBA Systems has also developed two programs for performing analytical triangulations with panoramic photography. The first program for Rome Air Development Center (RADC) included error models for the KC-56A rotating prism camera (RADC-TR-71-240) and the second for the Apollo Optical Bar Panoramic Camera (DBA, 1970). These programs were experimental in nature and not used in conventional triangulation.

2.8 Coordinate Determination

As indicated in the previous section the image space coordinates are a function of three basic classes of parameters: Exterior orientation, Systematic deformation parameters and Object Point. Typically the interior orientation parameters are calibrated prior to a block adjustment and are considered as known. The primary purpose of the block adjustment program is to derive the exterior orientation parameters and those object points carried in the block adjustment.

There are frequently many points not carried in the block adjustment for which object coordinates are desired. These coordinates are determined based on the intersection of rays from two or more photographic images of the same point. The same basic colinearity equations are used for this process except that the adjustment is only in terms of the object coordinates (X_j, Y_j, Z_j).

It should also be noted that the derived positions are in a rectangular coordinate system, therefore, during the last phase of the point positioning system these coordinates are converted to the desired system (UTM, LSR, USR, Geographic, etc.).

2.9 Data Base Concept

The concept of a photogrammetric point positioning data base is illustrated in Figure 2.0-1, assuming the material availability question is answered with a yes (In Sections 2.3 through 2.8 it was assumed the answer was no).If the material is available and if it has been previously adjusted it is only necessary to measure points on the photographs and perform the coordinate determination to derive object space position.

2.10 Summary

The preceding discussion of the various aero-triangulation processes are not meant to be all inclusive, but only to serve as an introduction to the ensuing analysis. Section 3.0 (Shortfall Analysis) provides a brief discussion of those areas of each process which DBA believes may need improvement. Section 4.0 (Discussion of Alternatives) describes a number of alternative techniques which will potentially overcome the shortfalls identified in Section 3.0. Lastly, Section 5.0 (Conclusions and Recommendations) discussed conclusions derived from the preceding sections and suggests a systematic approach for their implementation.

3.0 SHORTFALL ANALYSIS

The previous section outlined the current state-of-the-art in analytical aerotriangulation. This section presents a critical review of this system and identifies areas where improvements can be made in terms of (a) Accuracy, (b) Time, (c) Manual Labor involved and (d) Skill Level required. Each of the items discussed in the previous section will be analyzed for potential improvement in light of currently available technologies.

3.1 Material Availability

Current available material determinations are directly related to an individuals familiarity with past activities in an area. The amount of material, as well as quality may be directly related to the past activities of only one organization in an area and may not consider material available from other sources. Requirements may dictate the amount of time available for material research and consequently limit the magnitude of the material search.

The basic shortfall in determining material availability is the current inability to present to any user at any time all of the data pertaining to a specific area, in a timely manner.

3.2 Coverage Planning

Coverage planning,as currently employed, is generally a manual process. As with most manual operations, it is a tedious and time consuming effort. Consideration must be given to such parameters as scale of the final project, C-factor of the compilation instrument, type of acquisition sensor and platform, size and shape of the area of interest, etc. The time consumed during this process drastically reduces response time of the total Aero-triangulation System.

3.3 Material Acquisition

The process of material acquisition is directly related to the type of sensor used, incorporated with the type of platform available and, as previously mentioned, the processing of such material. Again the process is time consuming and related to the technologies of equipment and support materials rather than being an analytical problem. As such, DBA has no intent to delve further into this process.

3.4 Image Point Selection, Identification and Mensuration

During this process, a target is identified through photographic interpretation techniques, this step is normally performed manually by specially trained photointerpretors. Additionally, a target may be selected and identified on one type of photography and must be transferred very precisely onto a photographic image format from which measurements can be made. The point transfer and mensuration steps are normally performed manually with the aid of electro-optical measuring equipment. The total process is extremely time consuming as a result of the manual efforts required.

3.5 Data Preprocessing

There are three basic areas where steps can be taken in the data preprocessing which will minimize the computer run time for subsequent block adjustments. These areas are:

1. Computation of initial approximations,
2. Data editing, and
3. Data arrangement.

Obviously if initial values of the parameters to be adjusted are close to the true values fewer iterations of the least squares process wil be required (hence less run time). The deviation of initial approximations for some parameters (such as exposure station positions) can be very tedious and is often a guess at best. To rectify this problem, techniques for computationally deriving initial approximations related to special types of photography have been implemented on computer assisted stereocomparators. However, these techniques have not been applied to aerial photography nor to monoscopic instrumentation or as a step in the preprocessing for manual comparators.

The second area ; data editing, has been only superficially addressed for aerial photography. As with determination of initial values, techniques for pre-adjustment data editing have been implemented on a computer assisted stereocomputer. These techniques could be modified and implemented on computer assisted monoscopic comparators or in a preprocessing program.

The last area which is insufficiently explored in the data preprocessing step is the rearrangement of the data for efficient formation and solution of the normal equations. This step in most cases is currently performed manually. This step is slow and tedious and frequently overlooked, resulting in excessively long run times for jobs which could be performed much more efficiently. This is particularly true with non-parallel flight lines and photography of widely varying scale.

In analyzing the automatic reordering techniques currently employed (Glaser, 1971), it is clear that they each suffer from one or more of the following limitations:

1. they are not completely universal in that they require a certain degree of regularity in the block and cannot efficiently accommodate photos of greatly varying scales or in randomly oriented cross-strips;

2. they are not completely automatic in that they require human intervention to perform a manual reordering of the input data or to assess the improvement or optimization achieved;

3. they employ algorithms that do not provide, either manually or automatically, an optimum minimization;

4. they are designed only for the more common systems of banded normal equations and cannot accommodate the more complex banded- bordered systems.

3.6 Block Adjustments

Current analytical adjustment techniques require improvements to facilitate computations in three broad areas. The first area is the excessive computer run time generated with current algorithms due to inefficient data handling algorithms or improper exploitation of existing algorithms. Another major contributing factor is the incomplete data preprocessing (Section 3.5).

The second major shortfall of current aerotriangulation block adjustment is that current formulations provide for limited accuracy. They do not adequately model systematic errors nor do they consider all the auxiliary information that is available.

The third major drawback is the high skill level required to perform a block adjustment. This is because of the complexity of the algorithms employed.

Each of the specific programs described in Section 2.7 have limitations on the number of photographs, number and type of error models constraints etc. This class of limitation is dependent upon the specific algorithms implemented in the computational equations. Specific techniques used by individual programs will be discussed in Section 4.0 as general methods which could be implemented in other programs to increase their capability.

3.7 Coordinate Determination

The speed and accuracy with which point coordinates may be determined is directly related to the speed and accuracy of the Block Adjustment and Triangulation Process. The limiting factor for this process is the turnaround time for Data Reduction at a specific facility. Most often, such factors as priorities and management enter into this process. The discussion of such factors is beyond the scope of this report.

3.8 Summary

The preceding paragraphs identified specific areas in the Analytical Aerotriangulation System which require investigation for improvement. Shortfalls have been identified resulting from: inefficient data handling, time consuming reduction models, and required manual processes. Some of these shortfalls have been identified as being beyond the scope of this report or currently under investigation through other RADC efforts and will not be discussed further. However, several problem areas have been identified as being worthy of discussion; Material Availability, Coverage Planning, Data Preprocessing, and Block Adjustment. Detailed discussions for each of these areas relating to alternative techniques are addressed in Section 4 of this report.

4.0 DISCUSSION OF ALTERNATIVES

Section 3.0 identifies those processes in the Aerotriangulation System which can be improved through the use of state-of-the-art computation technology and advanced algorithms. The specific areas of interest include: Material Availability, Coverage Planning, Auxiliary Sensors, Ground Control Data, Data Preprocessing, and Block Adjustment. Alternative approaches and techniques for each of these are presented in the following subsections.

4.1 Material Availability

The use of modern mini-computers combined with state-of-the-art data base technology can be applied to the problem of accessing the available material pertinent to an area. This approach would be an operator interactive system which would allow the user to interogate a data base containing information relating to the coverage available over an area. This data base would include algorithms which specify the status and quality of this coverage. The system would consist of a series of CRT stations communicating with a master data base facility.

Such a system has been implemented by the USGS and is currently operational. This system provides the user with a listing which identifies the type of imagery, quality, percent of cloud cover data, first frame center, scale, altitude, and limits of coverage. This system is located at the EROS Data Center in Sioux Falls, South Dakota. This master data center is accessed by remote terminals at several USGS sites.

This system as currently configured will not meet the requirements for RADC or DMA, however, it is a base for development of a system tailored to specific Air Force requirements.

4.2 Coverage Planning

The manual process of Coverage Planning can be improved by use of a graphics display interactive terminal. Such a system could display a map of the area to be photographed. The operator could then indicate to the computer the approximate location of control along with the characteristics of the sensor. The computer would then display a flight profile including a ground footprint for the planned mission. The system would then perform a simulated block adjustment and display the theoretical error contours to be generated by such a flight plan. The user would then adjust his coverage parameters, control specification or auxiliary data requirements to optimize the coverage so that maximum utilization of the photography could be obtained with minimal expense.

The envisioned system could then produce a hard copy of the flight plan for use by the aircraft navigator in acquiring the photographs.

Another use of this system would be to assist strike mission planners in developing flight paths between two points by including such things as terrain avoidance algorithms. For these advanced applications, a digital data base of the area would be required.

4.3 Auxiliary Sensors Onboard Aircraft

There are a number of auxiliary sensors which are either currently available or will be available in the near future. These sensors have the capabilty of providing an external source of aircraft position and/or attitude which could be used as a supplement to the photographic system. These auxiliary sensors provide two potential benefits for aerotriangulation. The first is an increase in accuracy and the second is a set of consistent initial approximations to the exterior orientation parameters. The ability to have a consistent set of initial approximations is valuable in that preadjustment editing of data is more reliable and the number of iterations of the least squares algorithms could also be reduced.

It should be noted that many of these auxiliary sensors provide consistent but systematically biased data. These systematic bases can, in some cases be modeled so that useful information can be gained from the sensors.

The Statoscope and Airborne Profile Recorder are examples of such auxiliary sensors.The indications of absolute altitude and absolute change in altitude are of insufficient accuracy to improve the accuracy of the overall triangulation due to systematic biases in the system. In general, the difference between photogrammetrically derived exposure stations and Statoscope altitude observations have a systematic set of residuals which can easily be modeled in the block adjustment. These parameters would be included in the border of the normal equations as they would be common to a large set of photographs (Section 4.6.3

presents a detailed discussion of the border and Section 4.6.1.4 outlines the observational equations generated by the use of Statoscope type data).

The second type of sensor is the high dynamic Global Positioning System (GPS). This system, as currently configured, uses simultaneous range observation of four satellites to compute positional data at any instant in time. There are twenty four satellites in the proposed GPS configuration of which from five to twelve will be visible at any time. The current design goal for positioning with the GPS high dynamics system are (SAMSO,1974):

Horizontal	8	Meters	CEP
Vertical	10	Meters	PE
Velocity	0.1	Knots	PE

There are three alternative techniques available for use of this data. The first is the brute force approach of inputting the observed aircraft position at the time of the exposure along with the accuracy estimates indicated above. In most cases of interest, data with this level accuracy will add very little to the overall accuracy of the block adjustment. However, as initial approximations, this data is about three to four times more accurate than that currently used for standard triangulation. This aspect alone would be beneficial from the preadjustment data editing and possible reduction of the number of iterations required. This technique will require no change to existing triangulation programs other than possibly adding a coordinate transformation step in the data processing.

The second method for using GPS positional data is through error modeling. The errors identified above will not be independent between exposure stations. That is, there will be systematic biases between the GPS derived positions and the photogrammetrically derived positions. A large portion of this error can be attributed to the ephemeris errors and the geometry of the specific satellite constellation being observed by the GPS receiver unit. This gives rise to the error model approach for using the GPS derived position as a part of the photogrammetric reduction. This error model technique employs the development of a mathematical model which describes the difference between the GPS positions and the photogrammetrically derived position. This error model would contain a few parameters which would be common to a large subset of photographs. These parameters would then be added to the border of the overall least squares adjustment. A mathematical model for this is discussed in Section 4.6.1.3.

A third alternative technique actually models the continuous motion of the aircraft in space using a spline technique. The GPS receiver can be modified to output positional data at a very high frequency. For purposes of this discussion, assume that GPS is capable of providing data at a rate of thirty times per second. This data would be used to determine the aircraft's position in space with the photogrammetric data being constrained to lie along this derived function with appropriate error model terms to model the systematic discrepancy between the photogrammetric and GPS positions. The actual model for this system is discussed in Section 4.6.2.5.

Another source of external data is an onboard attitude and attitude rate sensor. This data can be used much as was done with the GPS data discussed above. Since this data can be used by the same basic techniques as the GPS data (i.e. direct input, error modeling or functional models), no attempt has been made in this report to explicitly discuss external attitude data. However, one should bear in mind that the techniques discussed for the GPS data can be applied to externally derived attitude data.

4.4 Ground Control Data

The normal type of Ground Control Data used for block adjustment is the position of photoidentifiable objects. This control data can be in a variety of coordinate systems and derived from a number of sources. Conventional ground surveying is the most common source. However, in recent years, satellite doppler surveying has become an extremely attractive method for determining precise ground positions particularly when one considers that this system derives points relative to the reference ellipsoid,not the Geoid as is the case in normal surveying. This permits one to easily develop a function relating the local Geoid to the reference ellipsoid in the area of interest. This is accomplished by observing a few of the conventional surveyed points with doppler tracking devices such as JMR or GEOCIEVER and by empirically modeling the differences between the derived coordinates.

In addition to the three dimensional absolute control, there are other types of information which could be exploited in a block adjustment to increase the accuracy without expensive surveying. These are: (1) relative positioning constraints,

(2) inertial survey data and (3) geometric constraints. Each of these items are discussed briefly in the following sections.

4.4.1 Relative Positioning Constraints

There are a number of sources of control information which have been classified as relative positioning constraints, four of which are listed below:

1. Local Survey
2. Distance Data
3. Azimuth or Angle Data
4. Equal or Differential Elevation Data

Local survey data consists of point coordinates which have been determined very precisely by local survey but have not been tied into an absolute datum. Many such local surveys may be available within the area covered by the adjustment. These surveys are locally consistent,but are biased relative to each other and to a master datum. These biases can be accommodated by the incorporation of datum shift parameters in the border of the block adjustment. PLODS (Haag & Hodge 1972) is the only program to our knowledge which currently has provisions for accommodating these datum shifts.

The second type of control information which is available is the precisely measured distance between two points, the locations of which may be unknown. This type of data could be collected by using Geodiometer type distance measurements across large lakes or between mountain tops in remote areas. PLODS and SAPGO have provisions to accommodate distance constraints. SAPGO requires that both points be on a single photograph. This restriction is not in PLODS since the distance constraint is carried in the border. Section 4.6.2.1 describes the mathematics for the rigorous inclusion of distance constraints in the border of a conventional block adjustment.

The azimuth of a line is another source of relative control which has heretofore not been properly exploited in analytical aerotriangulations. As with distance constraints both PLODS and SAPGO have the ability to handle azimuth constraints. SAPGO is limited in the single photo constraint whereas PLODS utilizes the border for azimuth constraints. The mathematics for adding the azimuth constraints to an aerotriangulation reduction is presented in Section 4.6.2.2

The equal and/or differential elevation constraints are also available in SAPGO and PLODS. DBA has also included this capability in our proprietory block adjustment-COMBAT-II. This constraint is generated by the fact that frequently points are known to have equal elevations, however, the value of that elevation is unknown. An example of this type of information is the points along the shoreline of a lake. The method by which this data has been included in PLODS and COMBAT-II is described in Section 4.6.2.3 of this report.

4.4.2 Inertial Survey Data

Within the past two years, new types of ground survey system have appeared on the market. These are the inertial surveying systems such as the Position and Azimuth Determination Systems (PADS) and the Litton Autosurveyor. These systems utilize acceleration observations derived from three orthogonal gyroscopes. These accelerations are nummerically integrated with respect to time by an onboard computer to produce velocity and to provide position data. This integration is performed every seventeen milliseconds.

The normal operation of the inertial surveying unit is to derive standard point positioning information. This type of object point data can be directly entered into an analytical triangulation as a control point. The accuracy of this data would be marginal in that the following are the system specifications for the (PADS)(ETL,1975);

Horizontal	20 Meters	CEP
Vertical	10 Meters	PE
Azimuth	1 Mil	RMS

However, accuracies of 1 meter horizontal CEP and .5 meters Vertical (PE) have been demonstrated with the Litton Autosurveyor by Gregarson in CANADA (Gregarson, 1975). This, however, is after adjustments have been made to known control with a systematic error distribution function applied. The basic model used by Gregarson is of the following form:

$$\Delta F - \Delta\phi = A\Delta\lambda + B\Delta\lambda t + E\Delta\phi + D$$

$$\Delta L - \Delta\lambda = A\Delta\phi + B\Delta\phi t + E\Delta\lambda + D$$

where F,L are the true latitude and longitude

ϕ,λ are the measured latitude and longitude

A- is the alignment error

B- is the drift in alignment

E- scale error of quantizer

D- error accumulator

t- time relative to start of tranverse

The Gregarson parameters (A, B, E, D) could easily be included in the analytical model for aerotriangulation.

Another approach to including inertial surveying observations directly in an aerotriangulation is through the use of an autoregressive procedure. This technique has been included in DBA's Doppler Survey reduction program and was documented by Brown and Trotter in 1969. However, the use of the autoregressive model in aerotriangulation has not been fully exploited. The general concept for its implementation is presented in Section 4.6.4.

Another attractive method for exploiting the capability of the inertial survey system is in conjunction with the geometric constraints discussed in the next section.

4.4.3 Geometric Constraints

In the past, analytical aerotriangulation has used only specific object points as the basic reference. That is, the same objective point is measured on multiple photographs. In recent years programs have been developed which incorporate some specific relationships between points such as distance, azimuth and equal elevation constraints as discussed in Section 4.4.1.

Greve et al in 1976 describes a program called the *"Generalized Multi-Photo Analytical Block Adjustment Program"*. This program was designed to handle a large number of non-metric type photographic views of the same general object. This program includes algorithms for constrainting specified sets of object points to lie on one or more geometric surfaces. The types of surfaces which this program can accommodate are:

- co-circular points

- co-spherical points
- co-cylindrical points
- co-linear points
- points lying on parallel lines
- coplaner points
- points on two planes
- points at equal intervals on a line

Properly exploited geometric constraints may be of tremendous practical utility in the analytical reduction of digital photographs. This can be demonstrated by a simple hypothetical example. Assume we have two aerial photographs of a road. This road is known to be straight. Then all points measured along this road should be constrained to line on a straight line in object space. Assume further that the edge of the road is digitized on each photograph independently. This implies that the points measured on one photograph have no direct correspondence to points measured on the other photograph. In other words all points measured on each photograph must be considered from a classical viewpoint as single ray points. In classical aerotriangulation the only single ray points which contribute to the solution are control points.* It is highly desirable to develop algorithms whereby these two sets of single ray points could be constrained to lie along a straight line in object space. A proposed method for developing such an algorithm is discussed in Section 4.6.2.4.

In general, there are a number of geometric constraints of this type which can be exploited in analytical reductions. A

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* Single ray pass points are frequently carried in the solutions when sub-blocks are reduced for editing purposes. However, these points are weighted so that they do not affect the solution.

typical constraints are:

- Sections of Roads and railroads can be defined as;

  Horizontal straight lines or,

  Horizontal sections of a circular curve.*

  Vertical straight lines or,

  Vertical sections of parabolic curve.

- Intersecting roads can be constrained to the same elevation.

- Points lying on the perimeter of objects which have known geometric shapes (such as storage towers) can be constrained to lie on the appropriate mathematical surface.

The use of this type of constraint will become more important as larger focal length imaging systems come into use, particularly with regard to the digital imaging sensors.

4.5 Data Preprocessing

As stated in Section 2.6, Data Preprocessing can be divided into four basic functions: Transformation, Generation, Editing and Arrangement. Section 3.6 identified the shortfalls associated with the last three areas. In this section alternative tehcniques are outlined, which, if implemented could possibly alleviate some of these shortfalls.

4.5.1 Generation of Initial Approximations

In classical bundle adjustment aerotriangulation processes, there are two classes of parameters which must be approximated. These are the elements of exterior orientation and the gound point coordinates for all pass points (Initial

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* Spiral curves are sometimes used for turnpike interchanges but we can omit them from practical consideration at the present

values for control points are set equal to their observed value.)

As mentioned earlier, the problem of determining initial approximations to the exterior orientation is of primary importance. An attractive approach to this can be envisioned if one realizes that the normal equation is rather insensitive to the actual measurements and are quite sensitive to the general location on the photograph and the number of points on the photograph. The classical normal equation can be represented as

$$\delta = N^{-1} C = (B^T WB)^{-1} B^T W \varepsilon \qquad 4.5.1\text{-}1$$

In general, both the weight matrix (W) and the jacobian matrix (B) are insensitive to the actual observations, but the discrepancy vector is very sensitive to the observation. If one realizes this, the $(B^T WB)^{-1} B^T W$ portion of equation 4.5.1-1 could be precomputed based on nominal values for the parameters and a nominal image point distribution. Multiplying the initial discrepancy (ε) by the matrix $(B^T WB)^{-1} B^T W$ would then result in a set of corrections to the nominal values. These corrections would then be applied to the initial approximations for the rigorous least squares solution. It should be noted that if one considers the weight matrix as a constant diagonal matrix with equal diagonal elements, it can be factored out. The net result is that this nominal matrix is exactly the same as the "Design matrix" employed by Rauhala(1976). Rauhala has demonstrated that this design matrix, when interpreted properly, can be used to simplify some least squares problems and yield the same results as normal techniques. The use of this method is very attractive for computing initial exposure station values.

A practical method of implementing this technique could be the generation of a theoretical design matrix using 9 points in classical locations. As the points are read into the preprocessor, the 9 points closest to those used in the generation of the theoretical design matrix could be used to compute the initial exterior orientation approximations. Expanding this technique to derive object space coordinates may also be possible.

Another alternative is the use of a recursive algorithm for the computation of initial approximations. DBA has implemented such a system, on the TP-3/P1 computer assisted stereocomputer for a special type of photography.

In general terms this same algorithm would be implemented in an off-line data preprocessing program or on a monoscopic computer assisted comparator. The basic philosophy of this recursive technique is that, each time a point is read (either on the comparator or into a computer), its contribution to the exterior orientation is computed. This contribution is then tested (data editing) prior to adding it to the previous estimate of the initial value. Therefore, as each image point is input, the inital values are refined. The result of this step is that the initial parameter values which are input to the block adjustment are much closer to the true values and the data is more unlikely to contain blunders.

4.5.2 Data Editing

Data Editing can be divided into two basic categories - Manual and Automatic. Manual being that the user physically detects and eliminates the aberrant observation. Automatic means that the computational algorithms detect and handle these blunders.

However, all automatic editing must be subject to manual review.

4.5.2.1 Manual Editing Algorithms

The efficiency and to some extent the ability of an operator to manually edit data is directly related to the manner and amount of data presented to him. An attractive approach for manual editing involves the use of a CRT display interacting with a computer which is capable of editing the data and pictorially presenting the results of the editing.

There are basically two techniques for implementing this type of system. The first is through the use of a remote terminal attached to a large scale CPU which is also used to perform the least squares adjustment. The second technique is through the use of a mini-computer as a dedicated editing station.

In either case, the results of the preprocessing could be tabulated and/or graphically displayed. These displays should be optimized for convenience of the user from an analysis standpoint. That is, all of the results should be available for display. However, key items should be displayed so that the user is not required to go through many pages of numeric tabulations to detect a few blunders. This implies that the routines should have some type of blunder detection algorithms so that potential blunders can be displayed in a manner which is readily apparent to the user.

4.5.2.2 Automatic Editing

During the preprocessing step the major problem with automatic editing of pass points is that image measurement residuals are a function of the mean object space coordinates which in turn are a function of the image measurements. This inter-relationship makes is extremely difficult to detect blunders. A solution to this is available when points appear on more than two photographs. That is, if a point appears on three or more photographs in a block there are two or more,two ray combinations of intersections which can be analyzed. A pair-wise analysis of the intersections of the rays will provide editing insight into the quality of the other rays for this point. This analysis can be in terms of residuals or the computed object space coordinates.

Another potential technique for preprocessing editing is an analysis of the difference between observed object space coordinates of control points and their computed value based on the initial exterior orientation approximations. These differences could be used as the basis for determining systematic discrepancies in the approximations to the exterior orientation parameters.

Yet another potential technique for editing data and computing the initial approximations to the exterior orientation parameters is a point by point updating scheme, whereby each point is input to the program and its contribution to the exterior orientation parameters is computed. This change in orientation is then used as a criteria for editing (See Section 4.5.1).

4.5.3 Reordering

Section 2.6.1, discussed current techniques for reordering the parameters to be adjusted in an analytical aerotriangulation to minimize the bandwidth of the normal equations thereby increasing the speed and efficiency of the solution. The basic short comings of these techniques were then summarized in Section 3.6.2. In summary, current techniques are limited for one or more of the following reasons:

1) they are not completely universal,
2) they are not completely automatic,
3) they do not provide optimum minimization, or
4) they can not accommodate banded bordered systems.

The most advanced of the current photogrammetric reordering or optimization techniques is that used in program COMBAT. This advanced algorithm was discovered and adapted to photogrammetric application as part of an in-house investigation performed by DBA in 1971. At that time, a study was made of the latest research in bandwidth minimization being conducted in other fields, such as structural stress analysis, which also encounter sparce diagonally banded matrices. Several promising techniques (Elphingstone,1972), (Akyuz & Utku, 1968), (Rosen, 1968) were adapted to the photogrammetric application and evaluated for speed and correctness of minimization. The most promising of these was (Glazer & Saliba, 1971) subsequently incorporated into Program COMBAT.

Although significant improvements are yet possible in the speed and efficiency of this algorith, its primary limitation is its current inability to handle banded-bordered systems.

As regards future development and improvement of photogrammetric bandwidth minimization, it is highly recommended that continuing research of this be sponsored. The primary objectives of this research would be to:

a) improve the speed and efficiency of existing algorithms,

b) develop or adapt newer, faster more optimum algorithms for banded systems,

c) develop or adapt algorithms to banded-bordered systems.

4.6 Block Adjustments

A number of techniques are available for increasing the efficiency and accuracy of photogrammetric block adjustments. These have been grouped under the following broad headings for purposes of this discussion:

1) Error Modeling
2) Constraints
3) Banded-Bordered Structure
4) Autoregressive Modeling
5) Automatic Editing

The first two items include methods whereby additional information can be added to the standard photogrammetric reduction. Many of these methods result in a set of normal equations which have a "banded-bordered" structure. An efficient algorithm for solving special types of banded-bordered structures is presented in Section 4.6.3.

Section 4.6.4 describes an approach whereby serially correlated observations can be rigorously incorporated in a least squares adjustment. As will be shown, this technique provides an alternative method for handling some error sources which have heretofore been modeled by empirically derived polynomial or Fourier series.

The last area where major technological emphasis is required is in the automatic editing aspects of a photogrammetric block adjustment. Each of these five major areas are described in detail in the following paragraphs.

4.6.1 Error Modeling

Systematic errors have been classically handled in photogrammetry by applying corrections to the basic observations in the preprocessing phase. However, for ultra-precise photogrammetric triangulation and for using observational data gathered by some types of auxiliary sensors it is necessary to carry parameters which model systematic errors as an integral part of the triangulation reduction. Section 4.6.1.1 reviews the general concept of including systematic error model parameters in a photogrammetric block adjustment. This is followed in Section 4.6.1.2 through 4.6.1.4 with the development of specific error models for Anomalous Distortions, GPS Positioning errors and Statoscope Observational errors.

4.6.1.1 Modeling Systematic errors in a photogrammetric reduction.

4.6.1.1.1 General Formulation

The classical photogrammetric bundle adjustment is based upon the colinearity condition equations which can be represented functionally* as

$$[x,y]^T_{ij} = f_c(X^c_i,\ \phi^c_i,\ X_j;\ C) \qquad 4.6.1\text{-}1$$

where

i - represents information pertaining to the i-th photograph

j - represents information pertaining to the j-th object point.

$[x,y]_{ij}$ - denote the image space coordinates of of the j-th point on the i-th photograph.

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* Explicit definition of this set of equations can be found in the Manual of Photogrammetry.

X_i^C - is a three element vector of exposure station position for the i-th photograph.

ϕ_i^C - is a three element vector of orientation angles for the i-th photograph.

X_j - is a three element vector of object space coordinates for the j-th ground point:

C - is a three element vector of interior orientation parameters

The vector C in the above representation contains the camera focal length, and the x,y principal point coordinates. These elements are normally treated as constants in a photogrammetric reduction, hence they are separated from the parameters by a semi-colon.

The above expression is valid, only if the true value of all the parameters and the image coordinates are known, and that all systematic errors have been properly accounted for.

In reality observations for the image coordinates are subject to random error. By denoting observations by a 'o' superscript this relationship can be expressed as:

$$[x,y]^T_{ij} = [x,y]^{oT}_{ij} + [v_x, v_y]^T_{ij} \qquad 4.6.1\text{-}2$$

where the v's denote observational residuals. Substituting this expression into 4.6.1-1 results in the following model.

$$[x,y]^{oT}_{ij} + [v_x, v_y]^T_{ij} = f(X_i^C, \phi_i^C, X_j; C) \qquad 4.6.1\text{-}3$$

This represents the classical observational equations used in bundle adjustments where the vector X_i^C, ϕ_i^C and X_j represent the parameters to be determined by a least squares adjustment.

However, if there are uncompensated systematic errors, equation 4.6.1-2 is invalid and should be rewritten as follows.

$$[x,y]^{T}_{ij} = [x,y]^{OT}_{ij} + [v_x \; v_y]^{T} + g\;(P) \qquad 4.6.1\text{-}4$$

where g denotes a systematic error model with parameters P. The exact form of the error model depends on the physics of the system. (Specific examples will be demonstrated in Section 4.6.1.2 through 4.6.1.4.) The parameter vector P will contain the p parameters required to model the error. Substituting this new representation of the true coordinates into the projective equation results in

$$[x,y]^{OT}_{ij} + [v_x \; v_y]^{T}_{ij} + g(P) = f(X^{C}_{i},\; \phi^{C}_{i}, X_j; C) \qquad 4.6.1\text{-}5$$

which can be rewritten as

$$[x,y]^{OT}_{ij} + [v_x \; v_y]^{T}_{ij} = h(X^{C}_{i},\; \phi^{C}_{i},\; X_j,\; P; C) \qquad 4.6.1\text{-}6$$

Equation 4.6.1-6 now represents the basic photogrammetric observational equations.

Assuming that h is a non-linear function of all parameters (X^C_i, ϕ^C_i, X_j, P), an iterative technique is required to solve for the parameters. We shall linearize the basic observational equations about some arbitrary initial values of the parameters and about the observed image measurements. The true value for the parameters are given in equation 4.6.1-7, where the superscript 'oo' denotes the arbitrary initial approximations and δ represents corrections to these approximations.

$$X^c_i = [X^c_i]^{oo} + \delta^{X^c}_i \qquad 4.6.1\text{-}7\ a$$

$$\phi^c_i = [\phi^c_i]^{oo} + \delta^{\phi^c}_i \qquad b$$

$$X_j = [X_j]^{oo} + \delta^X_j \qquad c$$

$$P = [P]^{oo} + \delta^P \qquad d$$

Substituting equations 4.6.1-7 into 4.6.1-6 results in

$$[x,y]^{oT}_{ij} + [v_x\ v_y]^T_{ij} = h(X^{c^{oo}}_i + \delta_{X^c_i},\ \phi^{c^{oo}}_i + \delta_{\phi^c_i}, X^{oo}_i + \delta_{X_i}, P^{oo} + \delta_p; c) \qquad 4.6.1\text{-}8$$

or by simply rearranging terms the following function can be developed.

$$O = F(x^o_{ij} + v_{x_{ij}},\ y_{ij} + v_{y_{ij}}, X^{c^{oo}}_j + \delta_{X^c_j},\ \phi^{c^{oo}}_j + \delta_{\phi^c_j}, X^{oo}_j + \delta_{X_j}, P^{oo} + \delta_p; c) \qquad 4.6.1\text{-}9$$

Equation 4.6.1- 9 can now be approximated by the first two terms of the Taylor series. This linearization process results in the following matrix equation.

$$A_{ij}\, v_{ij} + \dot{B}^{X^c}_{ij}\, \delta^{X^c}_j + \dot{B}^{\phi^c}_{ij}\, \delta^{\phi^c}_j + \overset{o}{B}_{ij}\, \delta^P + \ddot{B}_{ij}\, \delta^X_{ij} + \epsilon_{ij} = 0 \qquad 4.6.1\text{-}10$$

where the following matrix definitions hold.

$$\underset{(2,2)}{A_{ij}} = \left[\partial F_{ij} / \partial x_{ij}, y_{ij}\right] \qquad 4.6.1\text{-}11\ a$$

$$\underset{(2,1)}{v_{ij}} = [v_x\ v_y]^T \qquad b$$

$$\underset{(2,3)}{\dot{B}_{ij}^{X^c}} = \left[\partial F_{ij}/\partial X_j^c\right] \qquad 4.6.1\text{-}11 \quad c$$

$$\underset{(2,3)}{\dot{B}_{ij}^{\phi^c}} = \left[\partial F_{ij}/\partial \phi_j^c\right] \qquad d$$

$$\underset{(2,p)}{\overset{\beta}{B}_{ij}} = \left[\partial F_{ij}/\partial P\right] \qquad e$$

$$\underset{(2,3)}{\ddot{B}_{ij}} = \left[\partial F_{ij}/\partial X_j\right] \qquad f$$

$$\underset{(2,1)}{\varepsilon} = \left[x_{ij}^o, y_{ij}^o\right]^T - F(X_i^{c^{oo}}, \phi_i^{c^{oo}}, X_j^{oo}, P^{oo}; C) \qquad g$$

$\delta_i^{X^c}$, $\delta_i^{\phi^c}$, δ_j^X, δ^P are defined by equation 4.6.1-7.

The dimensions under the matrices are defined by the fact that F is a two element vector function (one function for x and one for y) which results in the A and B matrices having 2 rows. The number of columns in each Jacobian matrix is determined by the number of elements in the parameter vector, i.e. X_i^c, ϕ_i^c, X_j each contain 3 elements whereas P contains p elements. Each of these partial derivative matrices is evaluated at the initial approximation to the parameters. The discrepency vector ε is the difference between the observed image coordinates and the function F evaluated at the approximations.

Note that equation 4.6.1-10 is linear with respect to the parameter changes (δ). Equation 4.6.1-10 represents the linearized observation equations for the j-th object point imaged on the i-th photograph. Assume that there are a total of m photographs and a total of n points in the photogrammetric block to be adjusted.

Also assume for the moment that all points are imaged on all photographs*. First we shall collect all the linearized observational equations generated by the j-th ground point. This set of equations is

$$
\begin{array}{llllllll}
A_{1j}v_{1j} & & & + B_{1j}^{X^c}\delta_1^{X^c} & & +B_{1j}^{\phi^c}\delta^{\phi^c} & + \mathring{B}_{1j}\delta^P + \ddot{B}_{1j}\delta_j^X + \epsilon_{1j} & = 0 \\
 & A_{2j}v_{2j} & & + B_{2j}^{X^c}\delta_2^{X^c} & & +B_{2j}^{\phi^c}\delta_2^{\phi^c} & + \mathring{B}_{2j}\delta^P + \ddot{B}_{2j}\delta_j^X + \epsilon_{2j} & = 0 \\
 & & A_{3j}v_{3j} & +B_{3j}^{X^c}\delta_3^{X^c} & & +B_{3j}^{\phi^c}\delta_3^{\phi^c} & + \mathring{B}_{2j}\delta^P + \ddot{B}_{3j}\delta_j^X + \epsilon_{3j} & = 0 \\
 & & \ddots & & \ddots & \ddots & \vdots & \\
 & & & A_n v_n + & +B_{mj}^{X^c}\delta_m^{X^c} & +B_{mj}^{\phi^c}\delta_m^{\phi^c} & + \mathring{B}_{mj}\delta^P + \ddot{B}_{mj}\delta_j^X + \epsilon_{mj} & = 0
\end{array}
\qquad 4.6.1\text{-}12
$$

The offset between the A's, B^{X^c}'s, etc. is easily seen if one considers that the partial derivatives of the image points on the i-th photograph with respect to the parameters for the (i+1)st photograph are zero. In general the derivatives are zero for all photographs other than the i-th photograph.

Equation 4.6.1-12 can be rewritten more compactly using matrix notations as

$$
A_j v_j + B_j^{X^c}\delta^{X^c} + B_j^{\phi^c}\delta^{\phi^c} + \mathring{B}_j\delta^P + \ddot{B}_j\delta_j^{X^c} + \epsilon_j = 0 \qquad 4.6.1\text{-}13
$$

where the following definitions are employed

$$
\underset{(2m,2m)}{A_j} = \begin{bmatrix} A_{1j} & & & & \bigcirc \\ & A_{2j} & & & \\ & & A_{3j} & & \\ & & & \ddots & \\ \bigcirc & & & & A_{mj} \end{bmatrix}; \qquad 4.6.1\text{-}14 \text{ a}
$$

---

* This second assumption simplifies the presentation of the mathematical development and will be dropped later in this report.

$$\underset{(2m,1)}{v_j} = \begin{bmatrix} v_{1j} \\ v_{2j} \\ v_{3j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_{mj} \end{bmatrix} \qquad 4.6.1\text{-}14 \quad b$$

$$\underset{(2m,2m)}{B_j^{x^c}} = \begin{bmatrix} B_{1j}^{x^c} & & & & \bigcirc \\ & B_{2j}^{x^c} & & & \\ & & B_{3j}^{x^c} & & \\ & & & \ddots & \\ \bigcirc & & & & B_{mj}^{x^c} \end{bmatrix} ; \qquad c$$

$$\underset{(2m,3m)}{B_j^{\phi^c}} = \begin{bmatrix} B_{1j}^{\phi^c} & & & & \\ & B_{2j}^{\phi^c} & & & \\ & & B_{3j}^{\phi^c} & & \\ & & & \ddots & \\ & & & & B_{mj}^{\phi^c} \end{bmatrix} ; \qquad d$$

$$\underset{(3m,1)}{\delta x^c} = \begin{bmatrix} \delta x_1^c \\ \delta x_2^c \\ \delta x_3^c \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \delta x_m^c \end{bmatrix} ; \qquad 4.6.1\text{-}14 \quad e$$

$$\underset{(3m,1)}{\delta \phi^c} = \begin{bmatrix} \delta \phi_1^c \\ \delta \phi_2^c \\ \delta \phi_3^c \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \delta \phi_m^c \end{bmatrix} ; \qquad f$$

$$\underset{(2m,p)}{\overset{\circ}{B}_j} = \begin{bmatrix} \overset{\circ}{B}_{1j} \\ \overset{\circ}{B}_{2j} \\ \overset{\circ}{B}_{3j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overset{\circ}{B}_{mj} \end{bmatrix} ; \qquad g$$

$$\underset{(2m,3)}{\ddot{B}_m} = \begin{vmatrix} \ddot{B}_{1j} \\ \ddot{B}_{2j} \\ \ddot{B}_{3j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{B}_{mj} \end{vmatrix} ; \qquad 4.6.1\text{-}14 \quad h$$

$$\underset{(2m,1)}{\varepsilon_{1j}} = \begin{vmatrix} \varepsilon_{1j} \\ \varepsilon_{2j} \\ \varepsilon_{3j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{mj} \end{vmatrix} ; \qquad i$$

δ^P and δ_j^X are defined in equation 4.6.1-7d and c respectively.

The next step is the collection of all linearized observation equations for the n object points. This set of equations is

$$\begin{aligned}
A_1 v_1 \quad & + B_1^{X^c}\delta^{X^c} + B^{\delta^c}\delta^{\phi^c} + \mathring{B}_{1j}\delta^P + \ddot{B}_1\delta_1^X + && && + \epsilon_1 = 0 \\
A_2 v_2 \quad & + B_2^{X^c}\delta^{X^c} + B_2^{\phi^c}\delta^{\phi^c} + \mathring{B}_{2j}\delta^P + && \ddot{B}_2\delta_2^X && + \epsilon_2 = 0 \\
A_3 v_3 \quad & + B_3^{X^c}\delta^{X^c} + B_3^{\phi^c}\delta^{\phi^c} + \mathring{B}_{3j}\delta^P + && \quad \ddot{B}_3\delta_3^X && + \epsilon_3 = 0 \\
\ddots \quad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots && \qquad \ddots && \quad \vdots \qquad \vdots \\
A_n v_n \quad & + B_n^{X^c}\delta^{X} + B_n^{\phi^c}\delta^{\phi^c} + \mathring{B}_{nj}\delta^P + && \qquad \ddot{B}_n\delta_n^X && + \epsilon_n = 0
\end{aligned} \qquad 4.6.1\text{-}15$$

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As with equation 4.6.1-12, this set of equations can be rewritten in matrix form as:

$$A_v + B^{x^c}\,\delta^{x^c} + B^{\phi^c}\,\delta^{\phi^c} + \overset{\rho}{B}\delta^{P} + \ddot{B}\delta^{x} + \varepsilon = 0 \qquad 4.6.1\text{-}16$$

with the following matrix definitions;

$$\underset{(2mn,2mn)}{A} = \begin{bmatrix} A_1 & & & & \bigcirc \\ & A_2 & & & \\ & & A_3 & & \\ & & & \ddots & \\ \bigcirc & & & & A_n \end{bmatrix} \qquad 4.6.1\text{-}17 \quad a$$

$$\underset{(2mn,3m)}{B^{x^c}} = \begin{bmatrix} B_1^{x^c} \\ B_2^{x^c} \\ B_3^{x^c} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B_n^{x^c} \end{bmatrix} \qquad b$$

$$\underset{(2mn,3m)}{B^{\phi^c}} = \begin{bmatrix} B_1^{\phi^c} \\ B_2^{\phi^c} \\ B_3^{\phi^c} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B_n^{\phi^c} \end{bmatrix} \qquad c$$

$$\underset{(2mn,p)}{\overset{\circ}{B}} = \begin{bmatrix} \overset{\circ}{B}_1 \\ \overset{\circ}{B}_2 \\ \overset{\circ}{B}_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overset{\circ}{B}_n \end{bmatrix} \qquad 4.6.1\text{-}17 \quad d$$

$$\underset{(2mn,3n)}{\ddot{B}} = \begin{bmatrix} \ddot{B}_1 & & & & \\ & \ddot{B}_2 & & & \\ & & \ddot{B}_3 & & \\ & & & \ddots & \\ & & & & \ddot{B}_n \end{bmatrix} \qquad e$$

$$\underset{(2mn,1)}{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \qquad f$$

$$\underset{(3m,1)}{\delta^x} = \begin{bmatrix} \delta^x_1 \\ \delta^x_2 \\ \delta^x_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \delta^x_n \end{bmatrix} \qquad g$$

$$\underset{(2mn,1)}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_n \end{bmatrix} \qquad 4.6.1\text{-}17 \quad h$$

δ^{x^c} and δ^{ϕ^c} are defined by equations 4.6.1-14 e and f respectively. δ^P is defined by equation 4.6.1-7d. The elements of the other matrices in equation 4.6.1-17 are all defined by equation 4.6.1-14. Note that there are 2mn equations and 2mn + 6 m + 3n + p unknowns (v's, δ^{x^c}'s, δ^{ϕ^c}'s, δ^x's, and δ^P's). Obviously there are more unknowns than there are equations so that there are an infinite number of solutions which will satisfy equation 4.6.1-16. The most desirable solution is the one which satisfies the linearized observation equation and simultaneously minimizes the sum of the squares of the weighted residuals, i.e.

$$s - v^T \, w \, v \rightarrow \text{minimum value} \qquad 4.6.1\text{-}18$$

It has been demonstrated by Brown 1955 that this minimization is accomplished by solving the following equations:

$$N \, \delta = c \qquad 4.6.1\text{-}19$$

where

$$N = B^T (A \Lambda A^T)^{-1} B \qquad 4.6.1\text{-}20$$

$$c = B^T (A \Lambda A^T)^{-1} \varepsilon \qquad 4.6.1\text{-}21$$

$$\delta = \begin{bmatrix} \delta^{X^c} \\ \delta^{\phi^c} \\ \delta^{P} \\ \delta^{X} \end{bmatrix} \qquad 4.6.1\text{-}22$$

$$B = \begin{bmatrix} B^{X^c} & B^{\phi^c} & \overset{o}{B} & \ddot{B} \end{bmatrix} \qquad 4.6.1\text{-}23$$

The covariance matrix (Λ) and weight matrix (w) are discussed in depth in Section 4.6.1.1.4 and the solution of equation 4.6.1-19 for the parameters δ is discussed in Section 4.6.3.

Before proceeding with the specific examples for error models, there are three additional areas which must be discussed. These are; (1) the rearrangement of parameters, (2) the introduction of parameter observations and (3) the weighting of observations. These areas are described in the following subsections.

4.6.1.1.2 Rearrangement of Parameters

In equations 4.6.1-11 through 4.6.1-23 the exposure station (X^c) position was carried separately from the orientation angle parameters (ϕ^c). Since these two classes of parameters are associated with a single photograph, it is frequently convenient to define a six element vector of exposure station parameters (E) which is

$$\underset{(6,1)}{E_i} = \begin{bmatrix} X^c \\ \hline \phi^c \end{bmatrix}_i \qquad 4.6.1\text{-}24$$

Substituting 4.6.1-24 into 4.6.1-10, with the usual definition of initial approximations, results in the following set of linearized observational equations for the j-th point imaged on the i-th photograph

$$A_{ij}v_{ij} + \dot{B}_{ij}\dot{\delta}_i + \mathring{B}_{ij}\mathring{\delta} + \ddot{B}_{ij}\ddot{\delta}_{ij} + \varepsilon_{ij} = 0 \qquad 4.6.1\text{-}25$$

where

$$\underset{(2,6)}{\dot{B}} = \left[\partial F_{ij}/\partial X^c \;\middle|\; \partial F_{ij}/\partial \phi^c \right] = \left[B^{X^c}_{ij} \;\middle|\; B^{\phi^c}_{ij} \right]$$

$$\dot{\delta}_i = \begin{bmatrix} \delta^{X^c}_i \\ \delta^{\phi^c}_i \end{bmatrix}$$

$$\mathring{\delta} = \delta^P; \quad \ddot{\delta}_j = \delta^X_j$$

and all other elements are identical to those defined by equation 4.6.1-11. The collection process proceeds as was described for equations 4.6.1-12 through 4.6.1-17 with the obvious definition substitutions. The net result is a linearized set of observational equations which can be represented in matrix terms as:

$$A\nu + \dot{\dot{B}}\delta + \mathring{\mathring{B}}\delta + \dddot{B}\delta + \varepsilon = 0 \qquad 4.6.1\text{-}26$$

This is the general form of linearized observational equations which one would normally utilize to represent a photogrammetric triangulation with systematic error modeling. However, in other cases, it is desirable to separate the position elements from the orientation elements as expressed by equation 4.6.1-16. During the discussions which follow in Sections 4.6.1.2 through 4.6.1.4 specific examples, will be disscussed, some of which can be better treated using the form represented by 4.6.1-16 and others by the form 4.6.1-26.

4.6.1.1.3 Parameter Observations

As discussed in Section 4.6.1.1.1 the observational equations are linearized about arbitrary initial approximations. It is reasonable to assume that in some cases these parameters can be directly or indirectly observed. The purpose of this section is to outline a procedure for including observational equations for parameters directly into the overall system of linearized observational equations prior to initiating the least squares adjustment.

The inclusion of directly observed parameters is a straightforward process which has been documented in many reports (Ref. Brown,et al 1965,Davis&Riding:1970,Haag&Hodge 1972). In general, it proceeds as follows. Assume that observations for the ground point coordinates are available. The true value for these coordinates is

$$X_j = X_j^o + V_j^x \qquad 4.6.1\text{-}27$$

where;

X_j is the three element vector representing the true position of the object point.

X_j^o denotes the observed value for these object space coordinates. (Notice the superscript 'o' is used to denote observed quantities)

V_j^x is the three element vector of residuals.

Equation 4.6.1-27 can be substituted directly into equation 4.6.1-7c as follows;

$$X_j^o + V_j^x = X_j^{oo} + \delta_j^x$$

which by appropriately rearranging terms becomes

$$V_j^X - \delta_j^X + \varepsilon^X = 0 \qquad 4.6.1\text{-}28$$

where $\varepsilon^X = X_i^o - X_j^{oo}$

There will be one such equation generated for each observed object space point. It is assumed herein that all n object points are observed(this assumption will be discarded later). The collection of all sets of equation 4.6.1-28 results in the following matrix equation;

$$V^X - \delta^X + \varepsilon^X = 0 \qquad 4.6.1\text{-}29$$

where

$$\underset{(3n,1)}{V^X} = \begin{bmatrix} V_1^X \\ V_2^X \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_n^X \end{bmatrix}$$

$$\underset{(3n,1)}{\varepsilon^X} = \begin{bmatrix} \varepsilon_1^X \\ \varepsilon_2^X \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_n^X \end{bmatrix}$$

and δ^X is identical to the definition in 4.6.1-17q.

The logical question arises as to why not simply define the arbitrary initial parameter approximation (X_j^{oo}) as the observed value? This obviously reduces the ε^X vector to zero. However, one must remember that the parameters will be iteratively solved. At the completion of each solution the δ vector is applied to the previous parameter approximations and the solution is again performed. This correction cannot be applied to the observation, therefore, after the first iteration the approximation used for the linerization processes will be different from the observation, hence the ε^X vector will be non-zero.

Equation 4.6.1-27 implies that only random errors are associated with the observation of the j-th point. This may not be the case. Just as with image measurement there may be systematic errors associated with the observation of the j-th point. These systematic errors will be common to many other (if not all) object space points. For each discussion it is assumed that the systematic errors are in fact common to all points and can be modeled by a function g' which has a set of parameters P' just as for the image point observations.

Equation 4.6.1-27 then becomes:

$$X_j = X_j^o + V_j^X + g'(P') \qquad 4.6.1\text{-}30$$

which when substituted into equation 4.6.1-7c becomes:

$$X_j^o + V_j^X + g'(P') = X_j^{oo} + \delta_j^X \qquad 4.6.1\text{-}31$$

Proceeding as before this results in a linearized set of observational equations for the j-th points which are

$$V_j^X + \overset{\circ}{B}{}_j' \, \delta^{P'} - \delta_j^X + \varepsilon_j = 0 \qquad 4.6.1\text{-}32$$

This set of equations can be directly added to the structure given by 4.6.1-16 as outlined below. First redefine the vector P to include the parameter P' i.e.

$$\bar{P} = \begin{bmatrix} P \\ P' \end{bmatrix} \qquad 4.6.1\text{-}33$$

The partial derivative matrix $\overset{\circ}{B}_{ij}$ must also be redefined to include the partials of the image measurement function (F) with respect to the parameters P'. These partials will be zero. So that $\overset{\circ}{B}$ becomes

$$\overset{\circ}{B}_{ij} = \left[\partial F_{ij} / \partial P \;\middle|\; \partial F_{ij} / \partial P' \right] = \left[\partial F_{ij} / \partial P \;\; 0 \right] \qquad 4.6.1\text{-}34$$

The reverse is true of $\overset{\circ}{B}{}_j'$ it must be augmented by the derivative of the object point equation with respect to the image point error model parameter (P) i.e. ;

$$\overset{\circ}{B}{}_j' = \left[\partial g_j' / \partial P \;\middle|\; \partial g_j' / \partial P' \right] = \left[0 \;\middle|\; \partial g_j' / \partial P' \right] \qquad 4.6.1\text{-}35$$

Using this new definition of $\overset{\circ}{B}{}_j'$ all object point observational equations can be collected so that the following matrix representation is valid.

$$\ddot{V} + \overset{\circ}{B} \overset{\circ}{\delta} - \ddot{\delta} + \ddot{\varepsilon} = 0 \qquad 4.6.1\text{-}36$$

where

$$\ddot{V}_{(3m,1)} = \begin{bmatrix} V_1^X \\ V_2^X \\ V_3^X \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_n^X \end{bmatrix} ; \quad \overset{o}{B}'_{(3m,p+p')} = \begin{bmatrix} \overset{o}{B}{}'_1 \\ \overset{o}{B}{}'_2 \\ \overset{o}{B}{}'_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overset{o}{B}{}'_n \end{bmatrix} ; \quad \ddot{\delta} = \begin{bmatrix} \delta_1^X \\ \delta_2^X \\ \delta_3^X \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \delta_m^X \end{bmatrix} \quad \ddot{\varepsilon} = \begin{bmatrix} \varepsilon_1^X \\ \varepsilon_2^X \\ \varepsilon_3^X \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_m^X \end{bmatrix}$$

$\overset{o}{\delta}$ is the error model parameter correction vectors defined by;

$$\overset{o}{\delta} = \begin{bmatrix} \delta^P \\ \hline \delta^{P'} \end{bmatrix} \qquad 4.6.1\text{-}37$$

Rewriting both equation 4.6.1-16 and 4.6.1-36 the matrix definitive for B in equation 4.6.1-20 becomes:

$$\begin{aligned} V + B^{X^C}\,\delta^{X^C} + B^{\phi^C}\,\delta^{\phi^C} &+ \overset{o}{B}\overset{o}{\delta} + \ddot{B}\ddot{\delta} + \varepsilon = 0 \\ \ddot{V} \qquad\qquad &+ \overset{o}{B}{}'\overset{o}{\delta} - \ddot{\delta} + \ddot{\varepsilon} = 0 \end{aligned} \qquad 4.6.1\text{-}38$$

$$B = \begin{bmatrix} B^{X^C} & B^{\phi^C} & \overset{o}{B} & \ddot{B} \\ 0 & 0 & \overset{o}{B}{}' & -I \end{bmatrix}$$

The ε vector in equation 4.6.1-21 must be augmented by $\ddot{\varepsilon}$. It should be noted here that the $\overset{o}{B}$ matrix has been redefined by equation 4.6.1-34.

Throughout this section consideration has been given to observations of object space points. This is presented only as an example of the technique by which direct observations of parameters can be included in a photogrammetric adjustment. In fact all parameters are treated just as the object points discussed herein.The net result is a matrix equation representing all observations as:

$$\bar{A}\bar{V} + \bar{B}\bar{\delta} + \bar{\varepsilon} = 0 \qquad 4.6.1\text{-}39$$

where

$$\bar{V} = \begin{bmatrix} V \\ \dot{V}^{X^C} \\ \dot{V}^{\phi^C} \\ \mathring{V} \\ \ddot{V} \end{bmatrix}; \quad \bar{\delta} = \begin{bmatrix} \delta^{X^C} \\ \delta^{\phi^C} \\ \delta^{P} \\ \delta^{X} \end{bmatrix}; \quad \bar{\varepsilon} = \begin{bmatrix} \varepsilon^{X^C} \\ \varepsilon^{\phi^C} \\ \varepsilon^{P} \\ \varepsilon^{X} \end{bmatrix}$$

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$$\bar{B} = \begin{bmatrix} B^{X^C} & B^{\phi^C} & \mathring{B} & B \\ -I & 0 & \mathring{B}^{X^C} & 0 \\ 0 & -I & \mathring{B}^{\phi^C} & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & \mathring{B}^{X} & -I \end{bmatrix}$$

In this case the error model parameter vector P is

$$P = \begin{bmatrix} P' \\ P^{X^C} \\ P^{\phi^C} \\ P^{X} \end{bmatrix}$$

with P' being the error model parameters associated with the image points.

P^{X^C} being the error model parameters associated with the exposure station parameters.

P^{ϕ^C} being the error model parameters associated with the orientation angles, and;

P^{X} being the error model parameters associated with the object space coordinates.

Equation 4.6.1-39 represents the general linearized form of the observational equations.

4.6.1.1.4 Weighting of Observations

Throughout this section have been discussions of observations and the making of assumptions concerning the observability of all parameters. All observations must include an estimate of their accuracy. The technique for utilizing these apriori accuracy estimates in a photogrammetric triangulation are discussed in this paragraph. A natural fall-out of this discussion is a convenient technique for dealing with the assumption made earlier.

First , the manner by which apriori accuracy estimates for image measurements are utilized in rigorous triangulation programs will be described. It shall be assumed throughout this section that all observations are independent (i.e. the measurement of two image points are not correlated.) Representing the appropriate covariance matrix for the measurements of the j-th object point on the i-th photograph by Λ the normal covariance matrix definition is obtained.

$$\Lambda_{ij} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{bmatrix} \qquad 4.6.1\text{-}40$$

where

σ_x^2, σ_y^2 denote the x and y variance of the image measurements.

$\sigma_{xy} = \sigma_{yx}$ denote the covariance between the x and y coordinates.

Normally the x and y accuracy estimates are equal and uncorrelated. $\sigma_x^2 = \sigma_y^2$ and $\sigma_{xy} = 0$. However, if transformations are applied to the image measurement prior to the adjustment, then the accuracy estimates must also be transformed. In many cases this transformation of coordinates causes the entire matrix (4.6.1-40) to fill in. For this reason, the apriori covariance matrix will be considered as a full 2x2 matrix .

Equation 4.6.1-19 indicated that the desired solution for δ using the weighted least squares involved the formation of normal equations (N) by the formula:

$$N = B^T (A \Lambda A^T)^{-1} B \qquad 4.6.1\text{-}41$$

If A is an identity matrix (which it normally is in photogrammetric adjustments) then N, becomes $B^T \Lambda^{-1} B$. It is convenient to define the weight matrix W as the inverse apriori covariance matrix i.e.,

$$W_{ij} = \Lambda_{ij}^{-1} \qquad 4.6.1\text{-}42$$

In Section 4.6.1.1.1 it was assumed that all points appeared on all photographs. This assumption was necessary to simplify the collection of the observational equations using standard matrix notations. Obviously this assumption is not true. If a point does not appear on a photograph, then the observation of this point can be considered as having an infinite variance. This implies that the weight matrix associated with this point is zero. Theoretically one can consider all the non-existent observational equations as having a zero weight matrix.*

Just as was done for the observational equations, the weight matrices for all image points viewing a given point are collected into a matrix W_j which is ;

$$W_j = \begin{bmatrix} W_{1j} & & & & & & & O \\ & W_{2j} & & & & & & \\ & & W_{3j} & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ O & & & & & & & W_{mj} \end{bmatrix} \qquad 4.6.1\text{-}43$$

and then for all n object space points as ;

$$W = \begin{bmatrix} W_1 & & & & & & & O \\ & W2 & & & & & & \\ & & W_3 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ O & & & & & & & W_n \end{bmatrix}, \qquad 4.6.1\text{-}44$$

---

* Computational algorithms do not employ this technique, they simply use only the actual observational equations.

In both of the above cases the independence of observations is implied by the use of zero covariance terms relating observations of one image point to the next. Section 4.6.4 shall describe an algorithm for handling serially correlated data which violates this basic assumption of observational independence.

As described in Section 4.6.1.1.2, the image measurements are not the only source of observational data to be considered in a photogrammetric adjustment. Since parameters are considered as observable quantities, they also have associated with them an apriori uncertainty. These apriori accuracy esitimates are handled exactly the same as those described above for the image points.

4.6.1.2 Anomalous Distortion Error Model

It has been demonstrated by Brown, 1968, that after compensating for calibrated radial and decentering distortions in aerial photographs, there remain systematic patterns to the image measurement residuals after a least squares adjustment. This pattern is often not readily apparent when looking at the residuals on a single photograph, but it does become very apparent when the residuals for a large number of photographs are plotted on a single frame. The exact source of these systematic errors has not been identified, however, one can assume that the primary contributors are platten unflatness and uncompensated film deformation. An approach to modeling this systematic error is through the use of general polynomials. Equation 4.6.1-4 states that the true value of an image coordinate is equal to the observation plus a random residual plus an error model function g.

To model anomalous distortion we may define g as follows:

$$g_x = a_o + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + \ldots$$

$$g_y = b_o + b_1 x \; b_2 y + b_3 xy + b_4 x^2 + b_5 y^2 + \ldots$$

4.6.1-44

The a's and b's are the parameters. The number of parameters required will depend on the order of the polynomial employed. The example described by Brown in 1968 indicated that a fifth order polynomial was the upper limit.

Instead of the polynomial approach outlined above it may be desirable to define the error model function as a general two dimensional Fourier series.

As described by Brown 1974, a model of platten unflatness could also be included in the general adjustment by defining g as follows;

$$g_x = \frac{x}{r}(c_1 x^2 + c_2 xy + c_3 y^2 + c_4 w^3 + c_5 x^2 y + c_6 xy^2 + c_7 y^3)$$

$$g_y = \frac{y}{r}(c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x^3 + c_5 x^2 y + c_6 xy^2 + c_7{}^3)$$

4.6.1-45

In this case the polynomial (with parameters $c_1, c_2, \ldots.$) models the unflatness of the platten. Again a Fourier series could be used in the place of the polynomial.

4.6.1.3 G.P.S. Positioning Model

As described in Section 4.3 the Global Positioning System instrumentation can be mounted on aircraft to provide aircraft positioning at specific time intervals. The current expectations are that this positioning data will have an apriori uncertainty of 8 meters horizontally and 10 meters vertically. This level of accuracy will not appreciably increase the accuracy of photogrammetric triangulations. However, it will reduce the time for performing an adjustment, because it provides better initial approximations and has many editing benefits.

These accuracy estimates are relative to an absolute reference system and a majority of the error budget is contributed by uncertainties in the orbital ephemeris of the GPS satellites. Since the same satellite configuration will be tracked by the GPS receiver for a period of time, these orbital uncertainties will be common to a large number of photographs. For purposes of this discussion, assume that the same satellites are tracked for all photographs in the block. The exposure station coordinates can then be represented by the following matrix equations:

$$X_i^c = X_i^{c^o} + V_i^c + g\,(P) \qquad 4.6.1\text{-}46$$

where the observation $X_i^{c^o}$ is the GPS observed position at the time of exposing the i-th photograph. The error model function g(P) will model the systematic differences between the GPS position and the photogrammetrically determined position. The exact form of this function can only be determined after experimental data has been collected, however, it is reasonable to expect that a low order polynomial for each component will be adequate.

The assumption of using the same satellite constellation throughout an entire photographic mission is probably not reasonable. This can be accommodated in a basic model by a defined series of error model functions (g), one for each constellation selected. The specific function to be applied for any given photograph will be determined by the time interval over which the constellation was tracked and the time of the exposure. This can be thought of as reinitializing the error model whenever the GPS receiver changes constellations.

4.6.1.4 Statoscope Observations

The Statoscope is a device which measures differential changes in elevation of an aircraft in flight. This instrument can be keyed to the aerial camera so that height differences are recorded at each exposure station. The absolute accuracy of these differences depends to a large extent on the barometric characteristics of the atmosphere. Assume that an aircraft is flying at a constant elevation but that the barometric pressure is changing along the flight path. The statoscope would record this as a change in aircraft height. In realty, the output of the statoscope is a function of barometric pressure changes and aircraft altitude changes. It is reasonable to assume that the contribution from changing barometric surfaces can be modeled using a low order polynomial (probably an offset term and a drift term). This can be included into the overall scheme by defining the error model function for the exposure station appropriately.

4.6.2 Constraints

During the Data Acquisition, Reduction and Block Adjustment Phases, additional parameters relevant to object space parameters may be made available through auxiliary data. Such data may infer specific tolerances to which we may wish to constrain or enforce relative point parameters. This section defines the development of algorithms for the inclusion of such constraints into the normal equations. This will result in a stronger solution while at the same time maintaining the block diagonal integrity of the required portions of the coefficient matrix.

4.6.2.1 Distance Constraint

The distance constraint allows one to enforce a given tolerance on the chord distance between two surface points. As an example let the USR coordinates of the first point be represented by (X_1, Y_1, Z_1) and the second point by (X_2, Y_2, Z_2). Thus the distance constraint may be represented by:

$$\nu_{d_{12}} = \sqrt{(X_1-X_2)^2 + (Y_1-Y_2)^2 + (Z_1-Z_2)^2} \qquad (4.6.2.1\text{-}1)$$

Let us now define the partial derivatives of this constraint with respect to the USR coordinates of the surface point; such that;

$$\frac{\partial \nu_{d_{12}}}{\partial (X_1, Y_1 Z_1)} = A_d = -\frac{\partial \nu_{d_{12}}}{\partial (X_2, Y_2, Z_2)} \qquad (4.6.2.1\text{-}2)$$

in which,

$$A_d = \left[\frac{X_1 - X_2}{\nu_{d_{12}}} , \frac{Y_1 - Y_2}{\nu_{d_{12}}} , \frac{Z_1 - Z_2}{\nu_{d_{12}}} \right] \qquad (4.6.2.1\text{-}3)$$

We may now derive the observational equation for distance constraints. First of all let us define:

$$\dot{u}_{c_1} = \frac{\partial v_{d_{12}}}{\partial (X_1, Y_1, Z_1)} \quad , \quad \dot{u}_{c_2} = \frac{\partial v_{d_{12}}}{\partial (X_2, Y_2, Z_2)} \quad , \qquad (4.6.2.1\text{-}4)$$

$$\varepsilon' = v^{o}_{d_{12}} - v^{oo}_{d_{12}}$$

in which $v^{o}_{d_{12}}$ are observed parameters and $v^{oo}_{d_{12}}$ are approximated parameters.

Thus the observational equations are;

$$v'_d = \dot{u}_{c_1} \ddot{\delta}_1 + \dot{u}_{c_2} \ddot{\delta}_2 - \varepsilon' \qquad (4.6.2.1\text{-}5)$$

which may be written

$$v' = \dot{u}\, \ddot{\delta} - \varepsilon' \qquad (4.6.2.1\text{-}6)$$

Let us now assume a typical set of normal equations of the form;

$$\delta = N^{-1} C \qquad (4.6.2.1\text{-}7a)$$

which may be expressed through partitioning by ;

$$\begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{N} & \bar{N} \\ \bar{N}^T & \ddot{N} \end{bmatrix}^{-1} \begin{bmatrix} \dot{C} \\ \ddot{C} \end{bmatrix} \qquad (4.6.2.1\text{-}7a)$$

where $\dot{N}$ may be of the form $\begin{bmatrix} N & \hat{N} \\ \hat{N}^T & \overset{o}{N} \end{bmatrix}$ and $\bar{N}$ may be of the form $\begin{bmatrix} \bar{N} \\ \bar{\bar{N}} \end{bmatrix}$

To introduce the distance constraint let us solve for N and C from equation (4.6.2.1-7a) in the form of;

$$N = B^T WB$$

$$C = B^T W\varepsilon$$

such that:

$$\ddot{\delta} = (\dot{u}^T \; w' \; \dot{u})^{-1} \; (\dot{u}^T \; w' \varepsilon') \qquad (4.6.2.1\text{-}8)$$

in which w' is the corresponding constraint weight matrix and equation (4.6.2.1-8) is the equation to be introduced into equation (4.6.2.1- 7b), and results in:

$$\begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{N} & \bar{N} \\ \bar{N}^T & (\dot{u}^T w' \dot{u}) + \ddot{N} \end{bmatrix}^{-1} \begin{bmatrix} \dot{c} \\ (\dot{u}^T w \varepsilon') + \ddot{c} \end{bmatrix} \qquad (4.6.2.1\text{-}9)$$

As distance constraints may be applied to any combination of points, the constraint matrix may have elements located in any position, thus destroying the block diagonality of the $\ddot{N}$ matrix. In order to maintain the diagonality of the $\ddot{N}$ matrix, for folding and solution, these constraints must be introduced via an alternative method. Let us introduce an additional set of unknown parameters δ' such that equation (4.6.2.1-9) becomes:

$$\begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \\ \delta' \end{bmatrix} = \left[\begin{array}{cc|c} \dot{N} & \bar{N} & 0 \\ \bar{N}^T & \ddot{N} & \dot{u}^T \\ \hline 0 & \dot{u} & -w'^{-1} \end{array}\right]^{-1} \begin{bmatrix} \dot{c} \\ \ddot{c} \\ \varepsilon' \end{bmatrix} \qquad (4.6.2.1\text{-}10)$$

Equation (4.6.2.1-10) may now be arranged as follows in order to regain the structure of equation (4.6.2.1-7b):

$$\begin{bmatrix} \dot{\delta} \\ \delta' \\ \hline \ddot{\delta} \end{bmatrix} \left[\begin{array}{cc|c} \dot{N} & 0 & \bar{N} \\ 0 & -w'^{-1} & \dot{u} \\ \hline \bar{N}^T & \dot{u}^T & \ddot{N} \end{array}\right] \begin{bmatrix} \dot{c} \\ e' \\ \hline \ddot{c} \end{bmatrix} \qquad (4.6.2.1\text{-}11)$$

Thus, the constraint effects are placed into the border of the banded-bordered system and the banded integrity of the $\ddot{N}$ matrix is retained.

4.6.2.2 Azimuth Constraint

Another type of constraint which one may wish to apply to ground point data is the Azimuth Constraint. Such a constraint states that the angle between a line from the first point (X_1, Y_1, Z_1) to a reference (eg. True North) and a line from the first point (X_1, Y_1, Z_1) to a second point $(X_2, Y_2 Z_2)$ is held to a given tolerance. This azimuth constraint may be represented by;

$$\nu_{a_{12}} = \mathrm{TAN}^{-1}\left(\frac{Y_1 - Y_2}{X_1 - X_2}\right) \qquad (4.6.2.2\text{-}1)$$

The partial derivatives of this constraint may now be defined with respect to the USR coordinates of the surface points such that;

$$\frac{\partial \nu_{a_{12}}}{\partial (X_1, Y_1, Z_1)} = A_a = - \frac{\partial \nu_{a_{12}}}{\partial (X_2, Y_2, Z_2)} \qquad (4.6.2.2\text{-}2)$$

in which;

$$A_a = \left(- \frac{\cos^2 \nu_{a_{12}}}{(X_1 - X_2)^2} (Y_1 - Y_2) \, , \, \frac{\cos^2 \nu_{a_{12}}}{(X_1 - X_2)} \, , \, 0 \right) \qquad (4.6.2.2\text{-}3)$$

The observational elements $(\dot{u}_{c_1}, \dot{u}_{c_2}, \varepsilon')$ in equation (4.6.2.1-11) may now be defined with respect to the azimuth constraint parameters. Following the identical development represented in equations (4.6.2.1-5) through (4.6.2.1-11) it can be shown that the azimuth constraints can also be placed in the border structure so as not to degrade the $\ddot{N}$ structure.

4.6.2.3 Elevation Constraint

One additional constraint which may be applied to ground points is the elevation constraint wherein the elevation difference between two points is maintained to a desired tolerance. Let us assume a first point with geographic coordinates (ϕ_1, λ_1, h_1) and USR coordinates (X_1, Y_1, Z_1) and a second point with geographic coordinates (ϕ_2, λ_2, h_2) and USR coordinates (X_2, Y_2, Z_2). The equal elevation constraint between these two points may be represented by;

$$\nu_{h_{12}} = h_1 - h_2 \qquad (4.6.2.3\text{-}1)$$

Again we define the partial derivatives of this constraint with respect to the USR coordinates to obtain;

$$\frac{\partial \nu_{h_{12}}}{\partial (X_1, Y_1, Z_1)} = (\cos\phi_1 \cos\lambda_1 \ , \ \cos\phi_1 \sin\lambda_1 \ , \ \sin\phi_1) \qquad (4.6.3.3\text{-}2)$$

$$\frac{\partial \nu_{h_{12}}}{\partial (X_2, Y_2, Z_2)} = - (\cos\phi_2 \cos\lambda_2 \ , \ \cos\phi_2 \sin\lambda_2 \ , \ \sin\phi_2)$$

Again, the observational elements $(\dot{u}_{c_1}, \dot{u}_{c_2}, \varepsilon')$ are defined in equation (4.6.2.1-4) with respect to the equal elevation constraint parameters. Similarly the development represented in equations (4.6.2.1-5) through 4.6.2.1-11) is followed to place these constraints in the border structure and again maintain the integrity of the $\ddot{N}$ matrix.

4.6.2.4 Geometric Constraints

As identified in Section 4.4.3, a valuable source of information on aerial photographs is the geometric constraints imposed by the imagery. There are two basic approaches to this problem. The first is the employment of identified object space points and constraining them to lie along specific mathematical surfaces. This is the approach used by Greve et.al. 1976 and Brown et.al. 1971.

The second approach to include Geometric constraints is based upon a technique developed by Brown, 1967 called *The Method of Continuous Traces*. The technique was developed for precise geodetic positioning using ballistic cameras tracking a satellite.

The basic concept is illustrated in Figure 4.6.2-1. Three ballistic cameras (1,2,3) are viewing a single satellite pass (c). On each photograph a number of points are measured along the trace. The same point is not necessarily measured on multiple photographs. All of the points measured on a photograph generate a ruled surface illustrated by S_1,S_2, and S_3 in the figure. These three surfaces must intersect along the actual trajectory of the satellite (c). In this application, the orientation of each photograph was known by stellar reduction and it was assumed that two of the stations were at known locations with the only unknowns being the exposure station coordinates of the third station.

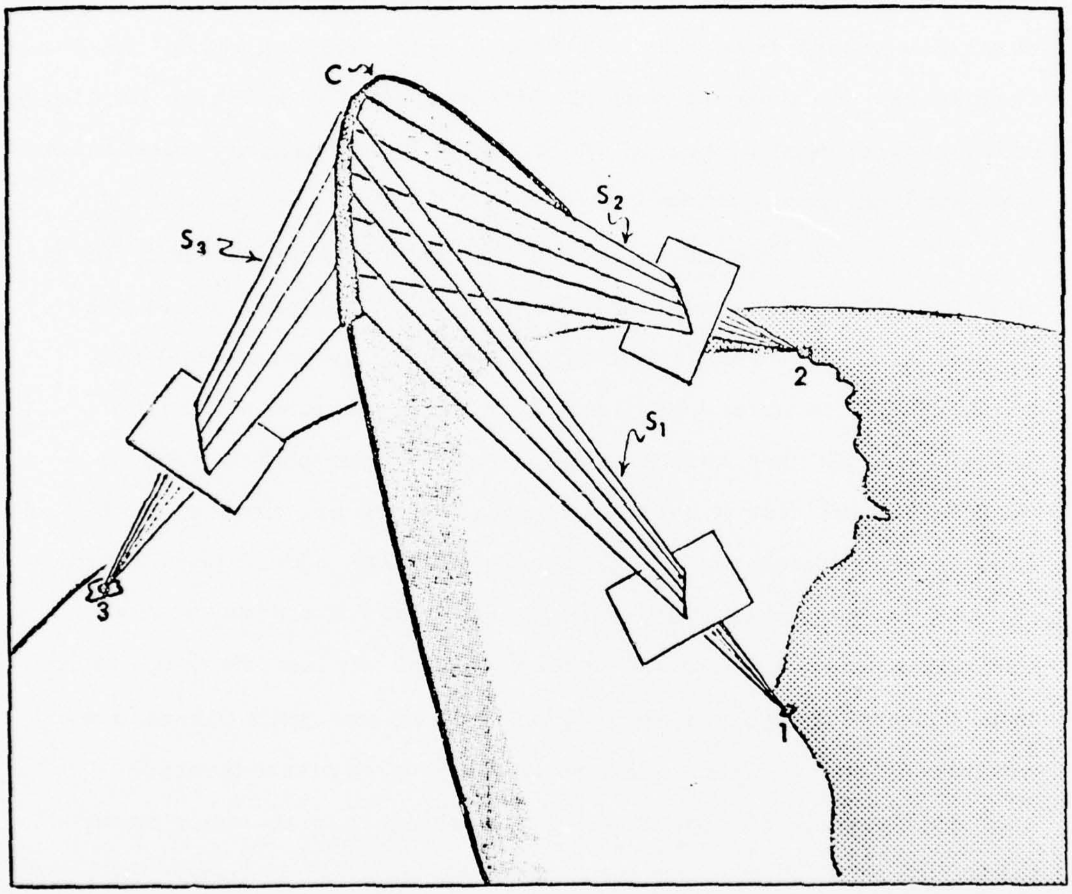

FIGURE 4.6.2-1

Illustrating Basic Principles of Method of Continuous Traces.

Brown demonstrated that a single arc was theoretically sufficient for complete determination of the point, however, this arc was only theoretical and not practical. However, if multiple satellite passes were observed from a given station, then, the three dimensional coordinates could be determined. The solution developed additional strength, as one would expect, when orbital constraints were applied to the curve C.

The idea of using the continuous trace concept for modeling geometric constraints in aerial photogrammetry is very attractive. The overall technique can best be understood by considering a simple case - namely a straight line in object space. Assume that this line is imaged on two photographs. Further assume that a series of measurements are made along the line on both photographs. In theory the line should be straight on both photographs. The measurements along the line on one photograph and the exposure station define a plane in space which must contain the object space line. Since two such planes are generated, one for each photograph, then their intersection will be the line in object space. Assuming that no other information is available , the intercepts of this line cannot be determined but its slopes can be. In fact, the intersecting planes will contribute directly to the determination of camera attitude, but not position. The additional fact that straight object space lines should be straight image space lines will also strengthen the recovery of some of the calibration or error model terms discussed in Section 4.6.1.

The concept of intersecting ruled surfaces can be expanded to include not only planes but generalized surfaces.

The advantage of the ruled surface concept over those heretofore implemented, lies in the ability to measure any number of points along an image on an individual photograph without transferring specific points to overlapping photographs.

Further advantages of this type of constraint may be realized through the use of slope data derived from an inertial surveying system such as PADS.

This type solution would also make better use of the available imagery on a photograph than is done by the classical point triangulation techniques.

4.6.2.5 Spline Constraints

The use of spline functions in photogrammetric triangulation to represent error model functions is very desirable in that a number of low order polynomials can be used to represent a highly erratic function. This is accomplished by forcing the polynomials to be continuous at the end point and to have equal first derivatives at the end points.

This type of function is potentially very useful in modeling the actual movement of an aircraft as a function of time. This type of constraint is an alternative method for incorporating the GPS data. Section 4.6.1 described a method whereby the GPS position at the time of each exposure could be included by modeling the difference between the

GPS position and the photogrammetrically derived position with a polynomial. In this section a procedure shall be outlined whereby the actual motion of the aircraft can be modeled.

This can best be explained by considering a single strip of photography. The time for acquiring this strip will start at $\bar{t}_o$ and continue until time $\bar{t}_k$. This time interval will be divided into k segments with end points and center points as illustrated below.

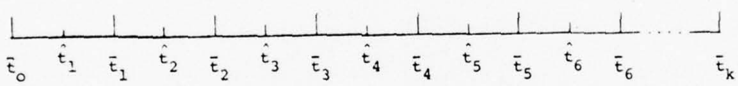

FIGURE 4.6.2-2

A series of third order polynomials will be used to represent the position of the aircraft during each time span $(\bar{t}_{\ell-1}<t<\bar{t}_{\ell}; \ell=1,2,3...k)$. The midpoint of each time span is $\hat{t}_{\ell}$ $(\ell=1,2...k)$. For each time span there will be three polynomials (one for each coordinate direction). For the X coordinate the following family of polynomials will be used.

$$S_1^{X^C}(t) = a_{11}^{X^C} + a_{12}^{X^C}(t-\hat{t}_1)+a_{13}^{X^C}(\hat{t}-t_1)^2+a_{14}^{X^C}(t-\hat{t}_1)^3; t_0 \le t < \bar{t}_1$$

$$S_2^{X^C}(t) = a_{21}^{X^C} + a_{22}^{X^C}(t-\hat{t}_2)+a_{23}^{X^C}(t-\hat{t}_2)^2+a_{24}^{X^C}(t-\hat{t}_2)^3; \bar{t}_1 \le t < \bar{t}_2$$

$$S_3^{X^C}(t) = a_{31}^{X^C} + a_{32}^{X^C}(t-\hat{t}_3)+a_{33}^{X^C}(t-\hat{t}_3)^2+a_{34}^{X^C}(t-\hat{t}_3)^3; \bar{t}_2 \le t < \bar{t}_3$$

.

. 4.6.2.5-1

.

.

$$S_k^{X^C}(t) = a_{k1}^{X^C} + a_{k2}^{X^C}(t-\hat{t}_k) + a_{k3}^{X^C}(t-\hat{t}_k)^2 + a_{k4}^{X^C}(t-\hat{t}_k); \bar{t}_{k-1} \le t < \bar{t}_k$$

In order to insure that the model is not discontinuous at the end points we shall enforce the constraints that the function S_ℓ and $S_{\ell+1}$ must have the same value at the common end point. Further, the aircraft velocity must be the same at that end point. These two constraints can be specified as:

End point continuity

$$S_1(\bar{t}_1) = S_2(\bar{t}_1)$$

$$S_2(\bar{t}_2) = S_3(\bar{t}_2) \qquad 4.6.2.5\text{-}2$$

.

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$$S_{k-1}(\bar{t}_{k-1}) = S_k(\bar{t}_{k-1})$$

Velocity Continuity

$$\dot{S}_1 (\bar{t}_1) = \dot{S}_2 (\bar{t}_{-1})$$

$$\dot{S}_2 (\bar{t}_2) = \dot{S}_3 (\bar{t}_2)$$

$$\vdots \qquad\qquad 4.6.2.5\text{-}3$$

$$\dot{S}_{k-1} (\bar{t}_{k-1}) = \dot{S}_k (\bar{t}_{k-1})$$

where

S_ℓ denotes the ℓ-th polynomial in any of the coordinate directions (X^c, Y^c, Z^c)

$\dot{S}_\ell$ denotes the time derivative of the ℓ-th polynomial in any of the coordinate directions

The process of enforcing equality of higher order time derivatives could also included if required. However, for our purposes the equal velocity constraint at the end points should be adequate. In general terms the ℓ-th polynomial can be written as

$$S_\ell(t) \begin{cases} a_{\ell 1} + a_{\ell 2}(t-\hat{t}_\ell) + a_{\ell 3}(t-\hat{t}_\ell)^2 + a_{\ell 4}(t-\hat{t}_\ell)^3, & \bar{t}_{\ell-1} \leq t \leq \bar{t}_\ell \, , \quad \hat{t}_\ell = \dfrac{\bar{t}_{\ell-1} + \bar{t}_\ell}{2} \\ 0 & \text{elsewhere.} \end{cases} \qquad 4.6.2.5\text{-}4$$

This equation is substituted directly into equation 4.6.1-1 for the vector X^c. This results in the basic functional form of our colinearity equations being:

$$[x,y]^T_{ij} = f_p(S(t_i),\ \phi^c_i,\ X_j;\ c) \qquad 4.6.2.5\text{-}5$$

where S represents the entire family of polynomial functions

and t_i denotes the time of exposing the i-th photograph.

Naturally S is a function of the parameter $a^{X^c}_{\ell 1}$, $a^{X^c}_{\ell 2}$, $a^{X^c}_{\ell 3}$, $a^{X^c}_{\ell 4}$, $a^{Y^c}_{\ell 1}$, ... $a^{Z^c}_{\ell 4}$. The direct observations of aircraft position from the GPS receiver are entered directly by the observation equations of the form 4.6.1-30 where the function g(P) is replaced with the family of polynomials S where B^{X^c} is defined as the derivative of S with respect to the parameters $a^g_{\ell s}$ ($\ell = 1,2,3 \ldots k$; $s = 1,2,3,4$; $g = X^c$, Y^c, Z^c).

Two sets of "dummy" observations are introduced to facilitate the incorporation of the spline continuity constraints. These dummy observation equations are:

$$\left.\begin{aligned} P_\ell &= S_{\ell-1}(\bar{t}_{\ell-1}) - S_\ell(\bar{t}_{\ell-1}) \\ D_\ell &= \dot{S}_{\ell-1}(\bar{t}_{\ell-1}) - \dot{S}_\ell(\bar{t}_{\ell-1}) \end{aligned}\right\} \quad \ell = 2,3,\ldots k \qquad 4.6.2.5\text{-}6$$

We are not really observing P or D, but because we desire to enforce the continuity constraints these values should be zero. We will therefore assign observations to them of zero and assign a very small apriori accuracy estimate. This effectively enforces the desired conditions. This approach to incorporating splines in a least squares adjustment was developed by DBA under a contract

AD-A038 138 DBA SYSTEMS INC MELBOURNE FLA F/G 14/5
STUDIES IN ADVANCED PHOTOGRAMMETRIC TRIANGULATION TECHNIQUES.(U)
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for ETL on the ability to perform triangulations with Side Looking Radar (DBA 1974).

The preceding paragraphs outline the method for introducing spline constraints into a photogrammetric adjustment. The following paragraphs will use GPS data as an example of how these constraints could be useful.

As was discussed earlier, the GPS receiver can be programmed to output positional data at any desired time. In Section 4.6.1.3 on Error Modeling, it was assumed that the receiver was triggered by an interrupt which was generated by the camera system to yield position at the time of the exposure.

An alternative to this ,is to record the GPS position data at the highest possible frequency. This recording would be accomplished completely independent of the photography process.The only requirement being that the time of each exposure also be recorded.

The GPS positional observations would be input directly into the photogrammetric adjustment, where they would affect only the recovery of the spline parameters. Since these spline parameters are included in the revised colinearity equations (Eq.4.6.2-5), the overall affect is to constrain the photographs to lie on the flight path defined by the continuous GPS data. If there are any systematic discrepancies between the GPS data and the photogrammetric data, they can be compensated for by the use of the standard error modeling techniques described in Section 4.6.1.

It is interesting to note that if one assumes equal length spline intervals and a uniform GPS sample rate, some computational savings can be made. This results from the fact that the normal equation contributions for the GPS are strictly a function of time, relative to the midpoint of the spline. With the above assumptions, it is apparent that the normal equations for this set of parameters will be constant and as such can be computed prior to the adjustment. This precomputed matrix can be added to the photogrammetric contribution of the normal equations during the formation algorith. This is not true of the correction vector (c) however. The GPS contribution must be computed for this vector because it is a function of time and the observational discrepancy vector. This principal of precomputing normal equations has been demonstrated by DBA 1976, in the *Offline Orthoprinter Software (OOPS)* development. This principal has also been employed by geodesists for years in least squares reductions. Geodesists compute normal equations once and reuse them for subsequent iterations updating only the constant column each time. In OOPS, a set of normal equations were set up and used for other data sets and recomputed only if the result indicated it was required.

With the technique outlined above, one would precompute only a portion of the normal equations and add it to the rigorously formed photogrammetric normals.

Earlier in this section an assumption was made that we would consider only a single strip of photography. The immediate question arises, "why not a block of photography?" There are two basic approaches to using splines for a block of photographs.
The first alternative is the reinitialization of the spline function for each strip, i.e. do not enforce the continuity constraints between the last spline of strip 1 and the first spline of strip 2, and do not use the GPS data for the time elapsed between strip 1 and 2.

The second alternative is to carry the spline functions around from one strip to the next. Obviously, there will be no photogrammetric data available between the end of strip 1 and the beginning of strip 2, however, the continuous GPS observations should be sufficient to derive the required parameters.

This discussion of the use of splines to model GPS data is only one possible application of splines in a photogrammetric type reduction. Two additional areas which could effectively use this type of modeling are; (1) the use of inertial surveying control information and (2) in conjunction with the geometric constraints described in Section 4.6.2.4.

4.6.3 Banded-Bordered Structure

The banded-bordered structure of the photogrammetric normal equation coefficient matrix is a developmental outgrowth of the strictly banded coefficient matrix. This development of a banded-bordered system was necessary to accommodate various classes of parameters previously not considered or parameters which would detract from the purely banded structure of the $\dot{N}$ or $\ddot{N}$ portions of the coefficient matrix. Classes of parameters are generally considered as those parameters relating to a specific group of observations. For example: GPS Data, as previously descibed, may be common to all photos in the block (Error Model Parameters, Section 4.6.1.3) or may be common to portions of strips or the entire strips (Spline Formation, Section 4.6.2.5). Miscellaneous parameters pertaining to certain sets of observations (i.e. Equal Elevation Constraints, Section 4.6.2.3) may also be placed in the border. Such information would normally detract from the banded structure of the N matrix by forming off-diagonal terms. With the subsequent development of recursive partitioning for solving banded-bordered systems, the maintenance of a minimum bandwidth is essential. Thus, it may be realized that a border will permit the handling of more varied data than previously used for the normal equation solution.

Throughout this section, examples of the banded-bordered structure will be presented utilizing algorithms previously developed for GPS Auxiliary Data (Sections 4.6.1 and 4.6.2). Subsection 4.6.3.1 will consider the GPS Error Modeling information as auxiliary data, whereas, Sections 4.6.3.2 (Structured Border) and 4.6.3.3 (Double Fold Solution Algorithms) will consider the GPS spline information as auxiliary data.

4.6.3.1 General Formulation of Banded-Bordered Normal Equations

In the formulation of a banded-bordered system, the initial geometric relationships between parameters are represented by analytical projection of arbitrary three-dimensional, object space coordinate system into a two-dimensional image space system. Considering the GPS Error Modeling Algorithms discussed in Section 4.6.1.3, the final projective equations expressed in matrix form are;

$$\nu + \dot{B}\,\dot{\delta} + \mathring{B}\,\mathring{\delta} + \ddot{B}\,\ddot{\delta} = \varepsilon \qquad (4.6.3\text{-}1)$$

where;

$\dot{B}\ \dot{\delta}$ - Photo parameter coefficients and correction

$\mathring{B}\ \mathring{\delta}$ - Error model coefficient and corrections

$\ddot{B}\ \ddot{\delta}$ - Object point coefficients and corrections

From equation (4.6.3-1) the following total set of observational equations may be derived;

$$\begin{aligned}
\nu + \dot{B}\dot{\delta} + \mathring{B}\,\mathring{\delta} + \ddot{B}\ddot{\delta} &= \varepsilon \\
\dot{\nu} - \dot{\delta} &= \dot{\varepsilon} \\
\mathring{\nu} - \mathring{\delta} &= \mathring{\varepsilon} \\
\ddot{\nu} - \ddot{\delta} &= \ddot{\varepsilon}
\end{aligned} \qquad (4.6.3\text{-}2)$$

which may be rewritten as;

$$\bar{v} + \bar{B}\,\bar{\delta} = \bar{\varepsilon}$$

where;

$$\bar{v} = \begin{bmatrix} \nu \\ \dot{\nu} \\ \mathring{\nu} \\ \ddot{\nu} \end{bmatrix} ; \quad \bar{B} = \begin{bmatrix} \dot{B} & \mathring{B} & \ddot{B} \\ -I & O & O \\ O & -I & O \\ O & O & -I \end{bmatrix} ; \quad \bar{\delta} = \begin{bmatrix} \dot{\delta} \\ \mathring{\delta} \\ \ddot{\delta} \end{bmatrix} ; \quad \bar{\varepsilon} = \begin{bmatrix} \varepsilon \\ \dot{\varepsilon} \\ \mathring{\varepsilon} \\ \ddot{\varepsilon} \end{bmatrix} \qquad (4.6.3\text{-}3)$$

In addition a diagonal weight matrix is formed;

$$w = \begin{bmatrix} w & & & 0 \\ & \dot{w} & & \\ & & \mathring{w} & \\ 0 & & & \ddot{w} \end{bmatrix}$$

Brown (1955) has shown that the least squares solution for the minimization of the variance is obtained when the vector δ is defined by a system of normal equations of the form:

$$\delta = N^{-1} c \qquad (4.6.3\text{-}4)$$

in which;

$$N = \bar{B}^T \bar{w} \bar{B} \qquad (4.6.3\text{-}5)$$

$$c = \bar{B}^T \bar{w} \bar{\varepsilon}$$

Substituting equation (4.6.3-3) into equations (4.6.3-5) we obtain;

$$N = \begin{bmatrix} \dot{B}^T & -I & 0 & 0 \\ \mathring{B}^T & 0 & -I & 0 \\ \ddot{B}^T & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} w & & & 0 \\ & \dot{w} & & \\ & & \mathring{w} & \\ 0 & & & \ddot{w} \end{bmatrix} \begin{bmatrix} \dot{B} & \mathring{B} & \ddot{B} \\ -I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}$$

$$N = \begin{bmatrix} \dot{B}^T w \dot{B} + \dot{w} & \dot{B}^T w \mathring{B} & \dot{B}^T w \ddot{B} \\ \mathring{B}^T w \dot{B} & \mathring{B}^T w \mathring{B} + \mathring{w} & \mathring{B}^T w \ddot{B} \\ \ddot{B}^T w \dot{B} & \ddot{B}^T w \mathring{B} & \ddot{B}^T w \ddot{B} + \ddot{w} \end{bmatrix} \qquad (4.6.3\text{-}6)$$

$$c = \begin{bmatrix} \dot{B}^T & -I & 0 & 0 \\ \mathring{B}^T & 0 & -I & 0 \\ \ddot{B}^T & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} w & & & 0 \\ & \dot{w} & & \\ & & \mathring{w} & \\ 0 & & & \ddot{w} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \dot{\varepsilon} \\ \mathring{\varepsilon} \\ \ddot{\varepsilon} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{B}^T w \varepsilon - \dot{w}\, \dot{\varepsilon} \\ \mathring{B}^T w \quad - \mathring{w}\, \mathring{\varepsilon} \\ \ddot{B}^T w \quad - \ddot{w}\, \ddot{\varepsilon} \end{bmatrix}$$

By redefinition and substitutions into equation (4.6.3-4), the normal equations become:

$$\begin{bmatrix} \dot{\delta} \\ \mathring{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{N} & \tilde{N} & \bar{N} \\ \tilde{N}^T & \mathring{N} & \tilde{\tilde{N}} \\ \bar{N}^T & \tilde{\tilde{N}}^T & \ddot{N} \end{bmatrix}^{-1} \begin{bmatrix} \dot{c} \\ \mathring{c} \\ \ddot{c} \end{bmatrix} \qquad (4.6.3\text{-}7)$$

wherein:

$\bar{N}$ - consists of the relationships between $\ddot{N}$ and $\dot{N}$,

$\tilde{N}$ - consists of the relationships between $\mathring{N}$ and $\dot{N}$, and

$\tilde{\tilde{N}}$ - consists of the relationships between $\ddot{N}$ and $\mathring{N}$.

The $\tilde{N}$, $\mathring{N}$, and $\tilde{N}^T$ terms constitute the border structure. An attempt must now be made to solve this sizable system of equations. In an effort to reduce the inversion effort of the large coefficient matrix a folding operation will be performed. This system of normal equations will first be partitioned and redefined as;

$$\begin{bmatrix} \dot{\delta}' \\ \hline \ddot{\delta} \end{bmatrix} = \left[\begin{array}{c|c} \dot{N}' & \bar{N}' \\ \hline \bar{N}^{T'} & \ddot{N} \end{array}\right]^{-1} \begin{bmatrix} \dot{c}' \\ \hline \ddot{c} \end{bmatrix} \qquad (4.6.3\text{-}8)$$

in which ; $\dot{\delta}' = \begin{bmatrix} \dot{\delta} \\ \mathring{\delta} \end{bmatrix}$; $\dot{N}' = \begin{bmatrix} \dot{N} & \tilde{N} \\ \tilde{N}^T & \mathring{N} \end{bmatrix}$; $\dot{c}' = \begin{bmatrix} \dot{c} \\ \mathring{c} \end{bmatrix}$; $\bar{N}' = \begin{bmatrix} \bar{N} \\ \tilde{\tilde{N}} \end{bmatrix}$

We shall now define the inverse of the normal equations to be;

$$N^{-1} = M' = \left[\begin{array}{c|c} \dot{M}' & \bar{N}' \\ \hline \bar{N}^{T'} & \ddot{N} \end{array}\right]^{-1} = \begin{bmatrix} \dot{M}' & \bar{M}' \\ \bar{M}^{T'} & \ddot{M}' \end{bmatrix} \qquad (4.6.3\text{-}9)$$

Since M'is the inverse of N by definition, we can say:

$$\begin{bmatrix} \dot{N}' & \bar{N}' \\ \bar{N}^{T'} & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{M}' & \bar{M}' \\ \bar{M}^{T'} & \ddot{M}' \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \qquad (4.6.3\text{-}10)$$

by conforming the required multiplication of the left hand side of equation (4.6.3-10) we obtain:

$$\begin{aligned} \dot{N}' \ \dot{M}' + \bar{N}' \ \bar{M}'^{T} &= I \\ \dot{N}' \ \bar{M}' + \bar{N}' \ \ddot{M}' &= O \\ \bar{N}^{T'} \ \dot{M}' + \ddot{N} \ \bar{M}'^{T} &= O \\ \bar{N}^{T} \ \bar{M}' + \ddot{N} \ \ddot{M}' &= I \end{aligned} \qquad (4.6.3\text{-}11)$$

from which we can solve for $\dot{M}$, $\bar{M}^{T}$, $\ddot{M}$ and obtain:

$$\dot{M}' = (\dot{N}' - \bar{N}'^{-1} \ddot{M} \bar{N}'^{T})^{-1} \qquad (4.6.3\text{-}12)$$

$$\bar{M}' = - \dot{M}' \ \bar{N}' \ \ddot{N}^{-1} \qquad (4.6.3\text{-}13)$$

$$\ddot{M}' = \ddot{N}^{-1} + \ddot{N}^{-1} \bar{N}^{T} \dot{M}' \ \bar{N} \ \ddot{N}^{-1} \qquad (4.6.3\text{-}14)$$

We may now define auxiliary forms in an effort to condense this formulation:

$$\text{Let us define: } \quad \begin{aligned} Q' &- \ddot{N}^{-1} \bar{N}'^{T} \\ R' &= \bar{N}' \ Q' \end{aligned} \qquad (4.6.3\text{-}15)$$

By substituting equations (4.6.3-12)through (4.6.3-14) into equation (4.6.3-8) and solving,we obtain:

$$\begin{aligned} \dot{\delta}' &= \dot{M}' (\dot{c}' - \bar{N}' \ \ddot{N}^{-1} \ \ddot{c}) \\ \ddot{\delta} &= \ddot{M}^{-1} \ \ddot{c} - \ddot{N}^{-1} \ \bar{M}'^{T} \ \dot{\delta} \end{aligned} \qquad (4.6.3\text{-}16a)$$

or with the use of auxiliaries;

$$\begin{aligned} \dot{\delta}' &= \dot{M}'(\dot{c}' - Q'\ddot{c}) \\ \ddot{\delta} &= \ddot{M}^{-1}\ddot{c} - Q' - \dot{\delta}' \end{aligned} \qquad (4.6.3\text{-}16b)$$

It should be noted that the largest inverse now required is the size of the $\dot{N}'$ portion of the coefficient matrix only and not the full coefficient matrix.

Figure 4.6.3-1 illustrates the structure of a typical set of normal equations generated during a photogrammetric block adjustment with error model parameters. Notice that in this example the $\bar{N}$ submatrix is highly patterned, however, the $\overset{o}{N}$, $\tilde{N}$ and $\tilde{\tilde{N}}$ are completely filled in. The next Figure (4.6.3-2) illustrates the structure which results from the first folding operation. This matrix is the $\dot{M}'^{-1}$ or $\dot{N}' \; -\bar{N}^{-1} \; (\ddot{N}+\ddot{W})^{-1}\bar{N}^{-T}$ which must be inverted. It should be noted here that the border is considered completely filled in. This structure is typical of the reduced normal equations in the orbitally constrained reductions LOSAT and PLODS. Haag and Hodge, 1972 describes a modification to the standard recursive partitioning algorithm to accommodate the border. This algorithm requires that the lower right block $(\overset{o}{N} - \tilde{\tilde{N}}\,(\ddot{N}+\ddot{W})^{-1}\,\tilde{\tilde{N}}^{T})$ be inverted and kept in core for the solution process. This restriction can become impractical when one considers the overall potential use of the border. The next section outlines an efficient algorithm for solving banded-bordered structures which have some specialized characteristics.

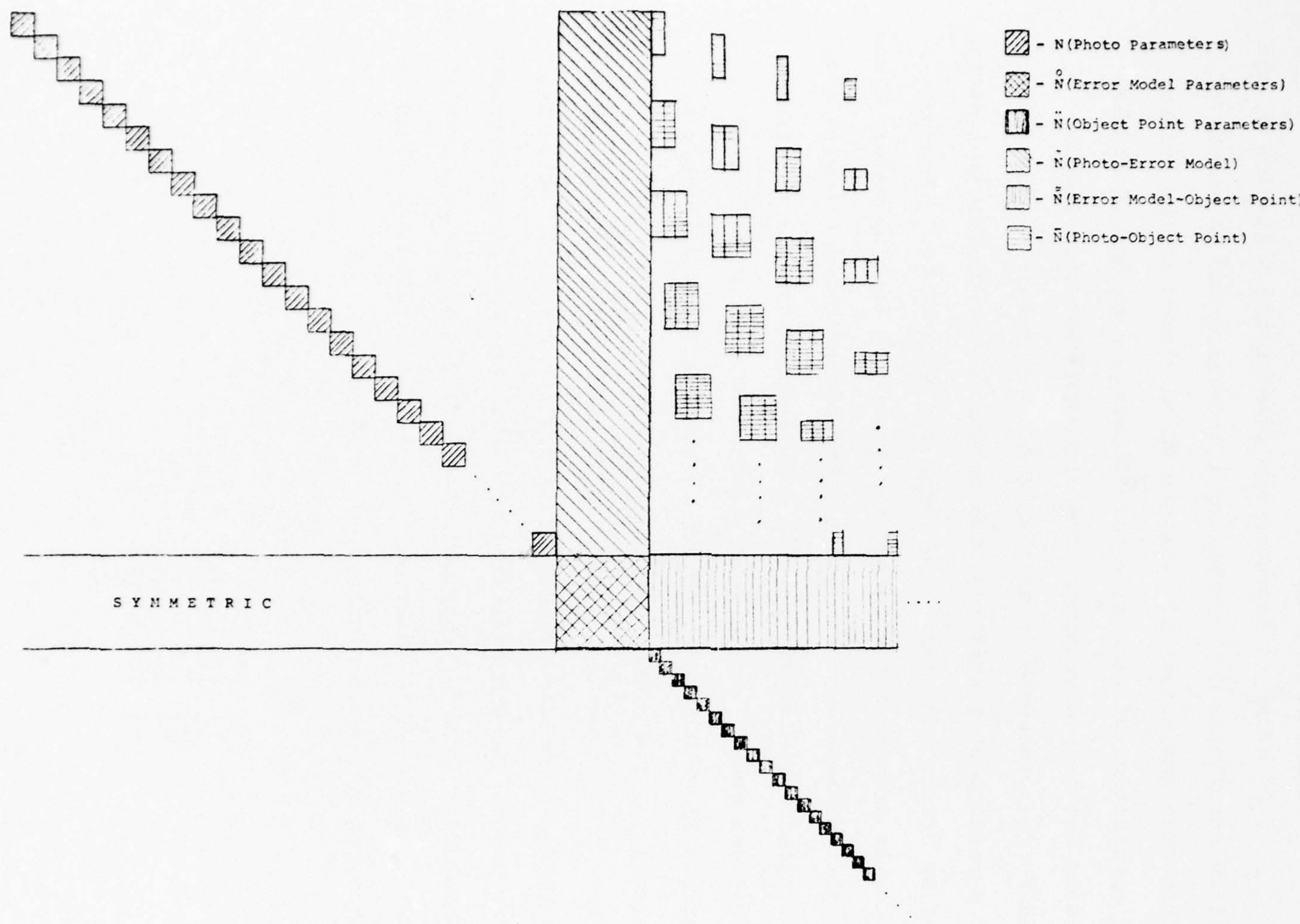

FIGURE 4.6.3-1
Normal Equations for a typical Four Photo Strip
(only upper triangulation portion is illustrated)

4.6.3.2 Structured Border

In order to decrease the size of the N portion of the coefficient matrix for conventional inversion and solution, or, to decrease the number of elements to be stored in the border for application of recursive partitioning, a method of double-folding has been developed. The algorithm for this double-folding are presented in Section 4.6.3.3. Before double-folding can be efficiently utilized, the border of the coefficient matrix must be structured similar to the $\tilde{N}$ and $\ddot{N}$ portions. In other words, an attempt to structure the $\mathring{N}$ portion in a symmetric block diagonal fashion must be made so that off-diagonal elements are eliminated. It is evident that only certain classes of parameters lend themselves to such structuring. In an effort to attain a block diagonal formation, the auxiliary data must be analyzed and the elements of the $\dot{N}$ portion if necessary to achieve this formation. Examples of such auxiliary data are the GPS spline algorithms developed in Section 4.6.2.5. Utilizing these algorithms and following the developmental format presented in Equations (4.6.3-4) through (4.6.3-7) the system of normal equations in the form of equation (4.6.3-7) is arrived at. This system can be double partitioned as follows:

$$\begin{bmatrix} \dot{\delta} \\ \hline \mathring{\delta} \\ \hline \ddot{\delta} \end{bmatrix} = \left[\begin{array}{c:c:c} \dot{N} & \tilde{N} & N \\ \hdashline N^T & \mathring{N} & \tilde{\tilde{N}} \\ \hdashline \bar{N}^T & \tilde{\tilde{N}}^T & \ddot{N} \end{array}\right]^{-1} \begin{bmatrix} \dot{C} \\ \hline \mathring{C} \\ \hline \ddot{C} \end{bmatrix}$$

An example of this double partitioned coefficient matrix with a structured border is shown in Figure 4.6.3-3. In addition to decreasing the storage required for the $\mathring{N}$ portion, one also reduces the computational effort required since only the block diagonal form need be computed.

4.6.3.3 Double Fold Solution Algorithms

With respect to Equation (4.6.3.8), the single folded algorithm effectively places the influence of the $\ddot{N}$ and $\bar{N}'$ portions of the coefficient matrix into the $\dot{N}'$ portion. Double folding will not place the effects of the once folded $\mathring{N}$ portion of Equation 4.6.3-7 into the $\dot{N}$ portion.

Again the following definitions are made which refer to Equation 4.6.3-7;

$$M = N^{-1} = \begin{bmatrix} \dot{N} & \tilde{N} & \bar{N} \\ \tilde{N}^T & \mathring{N} & \tilde{\tilde{N}} \\ \bar{N}^T & \tilde{\tilde{N}}^T & \ddot{N} \end{bmatrix}^{-1} = \begin{bmatrix} \dot{M} & \tilde{M} & M \\ \tilde{M}^T & \mathring{M} & \tilde{\tilde{M}} \\ \bar{M}^T & \tilde{\tilde{M}}^T & \ddot{M} \end{bmatrix} \qquad (4.6.3\text{-}17)$$

From equations 4.6.3-9 and 4.6.3-8 is derived;

$$\dot{M}' = \dot{N}'^{-1} = \begin{bmatrix} \dot{N} & \tilde{N} \\ \tilde{N}^T & \mathring{N} \end{bmatrix}^{-1} = \begin{bmatrix} \dot{M} & \tilde{M} \\ \tilde{M}^T & \mathring{M} \end{bmatrix} \qquad (4.6.3\text{-}18)$$

From equations 4.6.3-7 and 4.6.3-12 may be related the elements of equation 4.6.3-18 such that;

$$\dot{M}' = \begin{bmatrix} \dot{N} - \bar{N}\,\ddot{N}^{-1}\,\bar{N}^T & \tilde{N} - \bar{N}\,\ddot{N}\,\tilde{\tilde{N}}^T \\ \tilde{N}^T - \tilde{\tilde{N}}\,\ddot{N}^{-1}\,\bar{N}^T & \mathring{N} - \tilde{\tilde{N}}\,\ddot{N}^{-1}\tilde{\tilde{N}}^T \end{bmatrix}^{-1} \qquad (4.6.3\text{-}19)$$

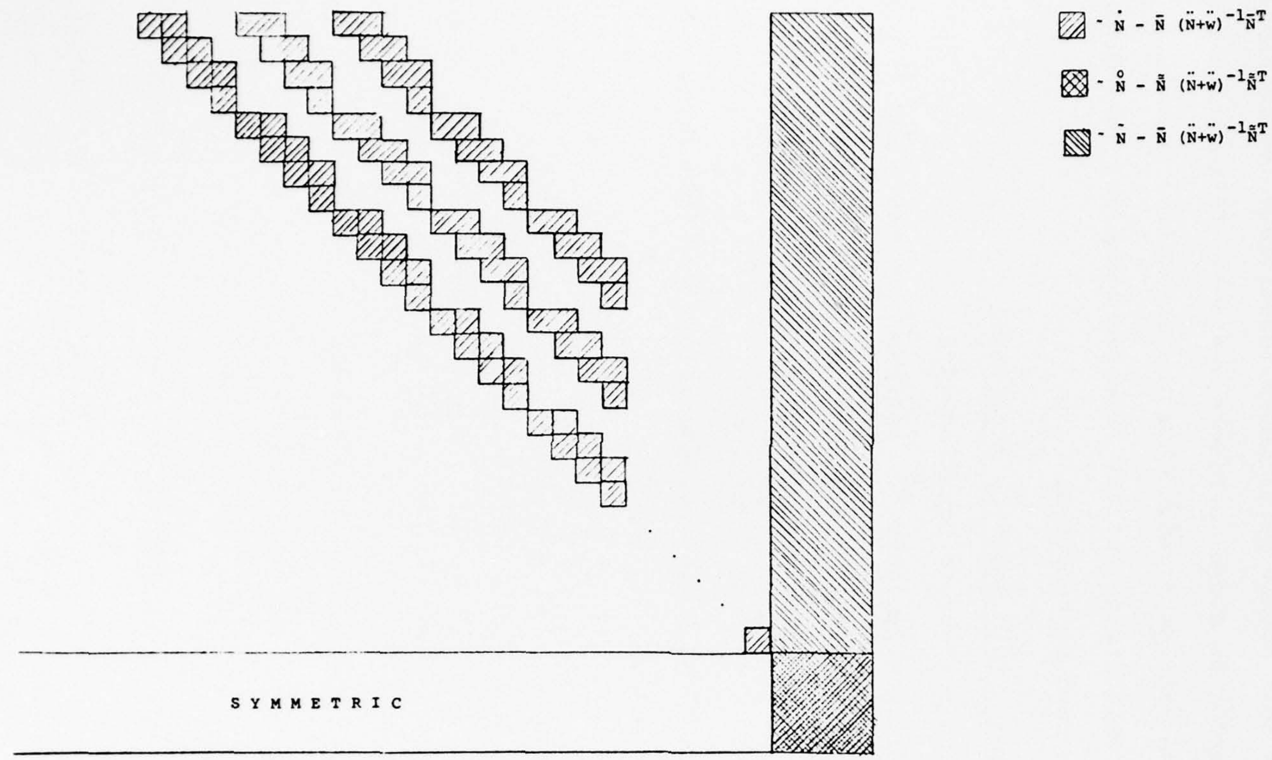

FIGURE 4.6.3-2
Reduced Normal Equations for a typical Four Photo Strip
(only upper triangular portion is illustrated)

By relating the elements in equation (4.6.3-19) to equation (4.6.3-9) and by substituting into equations (4.6.3-12) thru (4.6.3-14) the following is obtained;

$$\dot{M} = \left[(\dot{N}-\bar{N}\,\ddot{N}^{-1}\,\bar{N}^{T}) - (\tilde{N} - \bar{N}\,\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T})(\mathring{N}-\tilde{\tilde{N}}\,\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T})(\tilde{N}^{T}-\tilde{\tilde{N}}\,\ddot{N}^{-1}\bar{N}^{T})\right]^{-1}$$

$$\tilde{M} = -\,\dot{M}\,(\tilde{N} - \bar{N}\,\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T})(\dot{N} - \tilde{\tilde{N}}\,\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T})^{-1} \qquad (4.6.3\text{-}20)$$

$$\mathring{M} = (\mathring{N} - \tilde{\tilde{N}}\,\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T})^{-1} + (\mathring{N} - \tilde{\tilde{N}}\,\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T})^{-1}(\tilde{N}^{T}-\tilde{\tilde{N}}\,\ddot{N}^{-1}\bar{N}^{T})$$
$$\dot{M}(\tilde{N}-N\,\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T})(\mathring{N} - \tilde{\tilde{N}}\,\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T})^{-1}$$

The remaining elements of the inverted coefficient matrix may now be defined by;

$$\bar{N}' = \begin{bmatrix} \bar{N} \\ \tilde{\tilde{N}} \end{bmatrix} \qquad (4.6.3\text{-}21)$$

and by substituting equation 4.6.3-21 and 4.6.3-18 into equation 4.6.3-13

$$\bar{M}' = -\begin{bmatrix} \dot{M} & \tilde{M} \\ \tilde{M}^{T} & \mathring{M} \end{bmatrix} \begin{bmatrix} \bar{N} \\ \tilde{\tilde{N}} \end{bmatrix} \ddot{N}^{-1} = -\begin{bmatrix} (\dot{M}\bar{N} + \tilde{M}\,\tilde{\tilde{N}})\,\ddot{N}^{-1} \\ (\tilde{M}^{T}\,\bar{N} + \mathring{M}\,\tilde{\tilde{N}})\,\ddot{N}^{-1} \end{bmatrix} = \begin{bmatrix} \bar{M} \\ \tilde{\tilde{M}} \end{bmatrix} \qquad (4.6.3\text{-}22)$$

therefore;

$$\bar{M} = -\,(\dot{M}\,\bar{N}\,\ddot{N}^{-1} + \tilde{M}\,\tilde{\tilde{N}}\,\ddot{N}^{-1})$$
$$\tilde{\tilde{M}} = -\,(\tilde{M}^{T}\,\bar{N}\,\ddot{N}^{-1} + \mathring{M}\,\tilde{\tilde{N}}\,\ddot{N}^{-1}) \qquad (4.6.3\text{-}23)$$

Similarly, *equations 4.6.3-21 and 4.6.3-18 can be substituted into* equation 4.6.3-14 to obtain;

$$\ddot{M} = \ddot{N}^{-1} + \ddot{N}^{-1}\begin{bmatrix} \bar{N}^{T} & \tilde{\tilde{N}}^{T} \end{bmatrix}\begin{bmatrix} \dot{M} & \tilde{M} \\ \tilde{M}^{T} & \mathring{M} \end{bmatrix}\begin{bmatrix} \bar{N} \\ \tilde{\tilde{N}} \end{bmatrix}\ddot{N}^{-1}$$

$$= \ddot{N}^{-1} + \ddot{N}^{-1}\,\bar{N}^{T}\,\dot{M}\,\bar{N}\,\ddot{N}^{-1} + \ddot{N}^{-1}\,\tilde{\tilde{N}}^{T}\,\tilde{M}^{T}\,\bar{N}\,\ddot{N}^{-1} + \ddot{N}^{-1}\,\bar{N}^{T}\,\tilde{M}\,\tilde{\tilde{N}}\,\ddot{N}^{-1} +$$
$$\ddot{N}^{-1}\,\tilde{\tilde{N}}^{T}\,\mathring{M}\,\tilde{\tilde{N}}\,\ddot{N}^{-1} \qquad (4.6.3\text{-}24)$$

Relating elements from equations 4.6.3-20, 4.6.3-23 and 4.6.3-24 to corresponding elements in equations 4.6.3-12 through 4.6.3-14 a new set of auxiliary may be defined.

$$
\begin{aligned}
Q &= \ddot{N}^{-1} \bar{N}^{T} \\
Q &= \ddot{N}^{-1} \tilde{\tilde{N}}^{T} \\
R &= \bar{N} \ddot{N}^{-1} \bar{N}^{T} = \bar{N} Q \\
\tilde{R} &= \tilde{\tilde{N}} \ddot{N}^{-1} \tilde{\tilde{N}}^{T} = \tilde{\tilde{N}} \tilde{Q} \\
\tilde{\tilde{R}} &= \bar{N} \ddot{N}^{-1} \tilde{\tilde{N}}^{T} = \bar{N} \tilde{Q} \\
\mathring{Q} &= (\mathring{N} - \tilde{R})^{-1} (\tilde{N}^{T} - \tilde{\tilde{R}}^{T}) \\
\mathring{R} &= (\tilde{N} - \tilde{\tilde{R}}) (\mathring{N} - \tilde{R})^{-1} (\tilde{N} - \tilde{\tilde{R}})^{T} = (\tilde{N} - \tilde{\tilde{R}}) \mathring{Q}
\end{aligned}
\tag{4.6.3-25}
$$

By substituting into equations 4.6.3-20, 4.6.3-23 and 4.6.3-24 the equations can be simplified to be;

$$
\begin{aligned}
\dot{M} &= (\dot{N} - R - \mathring{R})^{-1} \\
\mathring{M} &= (\mathring{N} - \tilde{R})^{-1} + \mathring{Q} \dot{M} \mathring{Q}^{T} \\
\tilde{M} &= - \dot{M} \mathring{Q}^{T} \\
\bar{M} &= - (\dot{M} Q^{T} + \tilde{M} \tilde{Q}^{T}) \\
\tilde{\tilde{M}} &= - (\tilde{M}^{T} Q^{T} + \mathring{M} \tilde{Q}^{T}) \\
\ddot{M} &= \ddot{N}^{-1} + \tilde{Q} (\mathring{N} - \tilde{R})^{-1} \tilde{Q}^{T} - (Q - \tilde{Q} \mathring{Q}) \dot{M} (Q - \tilde{Q} \mathring{Q})^{T}
\end{aligned}
\tag{4.6.3-26}
$$

The normal equations may now be defined as;

$$\delta = N^{-1}$$

or

$$\begin{bmatrix} \dot{\delta} \\ \mathring{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{M} & \tilde{M} & \bar{M} \\ \tilde{M}^T & \mathring{M} & \tilde{\tilde{M}} \\ \bar{M}^T & \tilde{\tilde{M}}^T & \ddot{M} \end{bmatrix} \begin{bmatrix} \dot{c} \\ \mathring{c} \\ \ddot{c} \end{bmatrix} \qquad (4.6.3\text{-}27)$$

By substituting equations (4.6.3-26) into the normal equations (4.6.3-27) and solving, the following is obtained;

$$\begin{aligned} \dot{\delta} &= \dot{M}(\dot{c} - \mathring{Q}^T \mathring{c} - \ddot{c}\,(Q^T - \mathring{Q}^T \tilde{Q}^T)) \\ \mathring{\delta} &= (\mathring{N} - \tilde{R})^{-1}\,(\mathring{c} - \tilde{Q}^T \ddot{c}) - \mathring{Q}\,\dot{\delta} \\ \ddot{\delta} &= \ddot{N}^{-1}\,\ddot{c} - Q\,\dot{\delta} - \tilde{Q}\,\mathring{\delta} \end{aligned} \qquad (4.6.3\text{-}28)$$

The result of this second folding is that the largest matrix now required for conventional inversion is the $\dot{N}$ or $\mathring{N}$ portions of the coefficient matrix. To illustrate the potential utility of this algorithm, two cases of GPS data with the spline modeling described in Section 4.6.2.5 shall be considered. The first case will have the splines in the border and in the second case, the photo orientation elements will be in the border with the splines in the $\dot{N}$ position.

Figure 4.6.3-3 illustrates the normal equations generated by a typical four strip block of photographs where each strip contains 20 photographs. The time span for each spline is arbitrarily defined as five photographs in length. The $\bar{N}$ or $\tilde{\tilde{N}}$ matrices are not completely illustrated but they do represent the worst case condition.

The $\dot{N}$ submatrices represent the photo orientation angles arranged by the standard cross strip numbering scheme. $\mathring{N}$ is the submatrix generated by the spline coefficients. There are basically three elements to this matrix which are illustrated by the numbers 1, 2, and 3 in the figure. The first is the contribution to the spline parameters by the GPS data (1) in the figure. Elements 2 (in the figure) are the contribution to the spline coefficients from the photogrammetric exterior orientation parameters and elements. The third element (in the figure) represents the contribution from the spline continuity constraints. Elements 1 and 3 are those which were previously discussed for potential precomputing of their values.

It is important to notice the highly patterned structure of the $\tilde{N}$, $\bar{N}$ and $\tilde{\tilde{N}}$ matrices which is in contrast to the discussion in Section 4.6.3-1.

Figure 4.6.3-4 illustrates the results of tne first folding operation when, the $\ddot{N}$ portion is folded into the $\dot{N}$, $\mathring{N}$ $\tilde{N}$ submatrices. Notice again that inthis figure, $\tilde{N}$ is still highly patterned and $\mathring{N}$ has a banded structure.

The second folding ($\mathring{N}$ into $\dot{N}$) requires the inverse of the matrices ($\mathring{N}-\tilde{R}$) illustrated as the lower right block of Figure 4.6.3-4. Naturally, this inversion process will cause the matrix to be completely filled, however, if one can assume that the elements beyond the bandwidth are insignificant numerically, then the inverse will have the same essential structure as the normal equations*. Based on this assumption, the results of this

---

* This assumption may well be invalid and requires further testing.

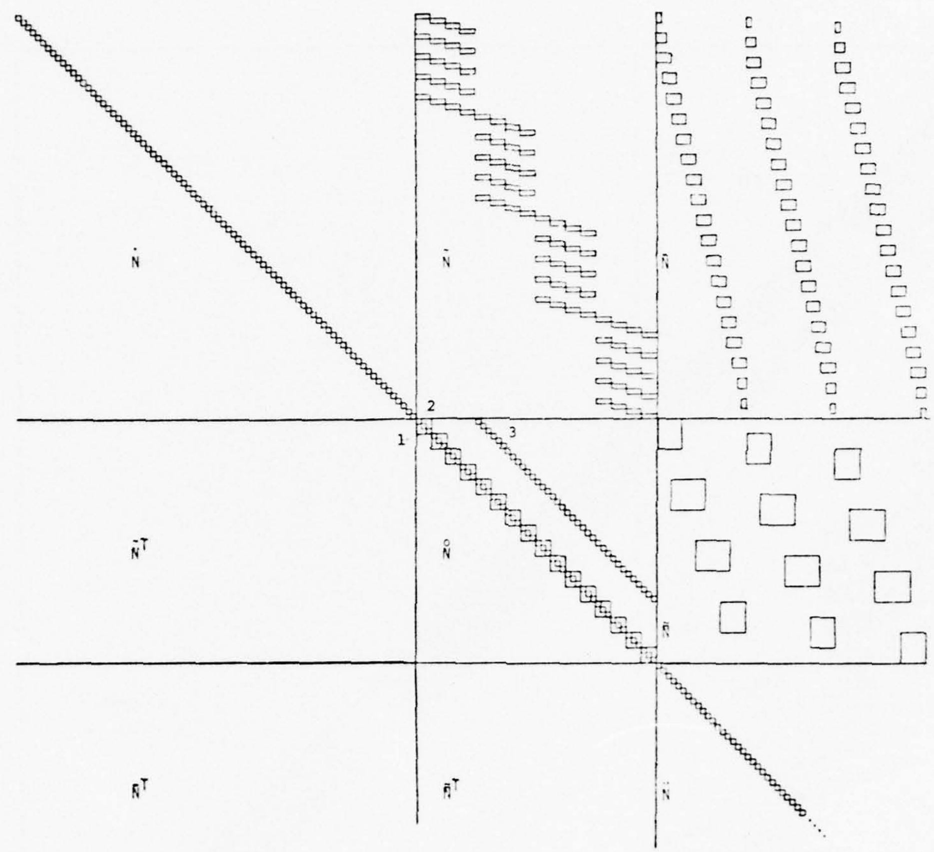

FIGURE 4.6.3-3
Typical Normal Equations for a 4x20 Photo Block with
Five Photo Splines

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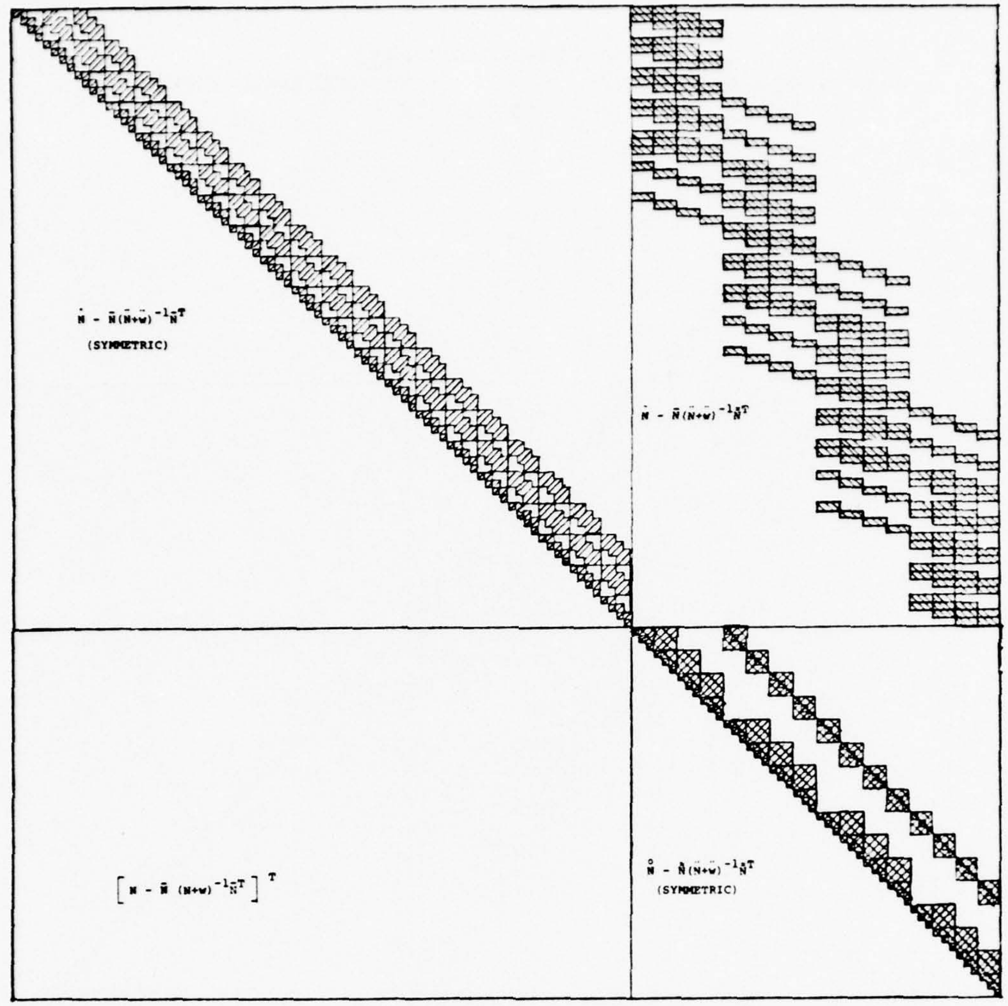

FIGURE 4.6.3-4
Results of the First Fold on the Normal Equations
(Case I)

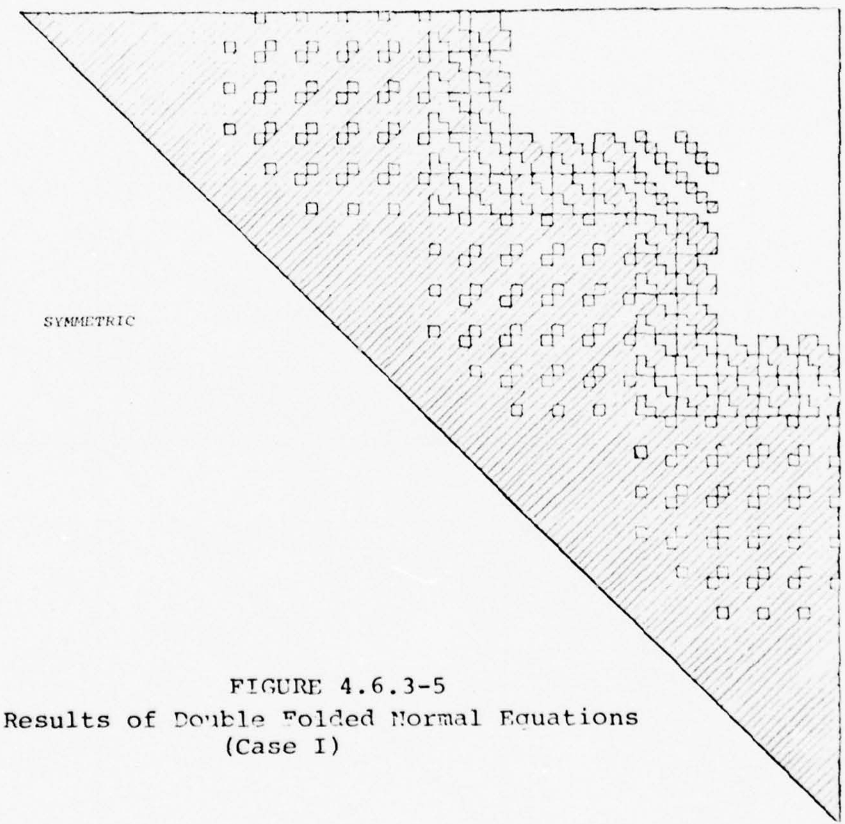

FIGURE 4.6.3-5
Results of Double Folded Normal Equations
(Case I)

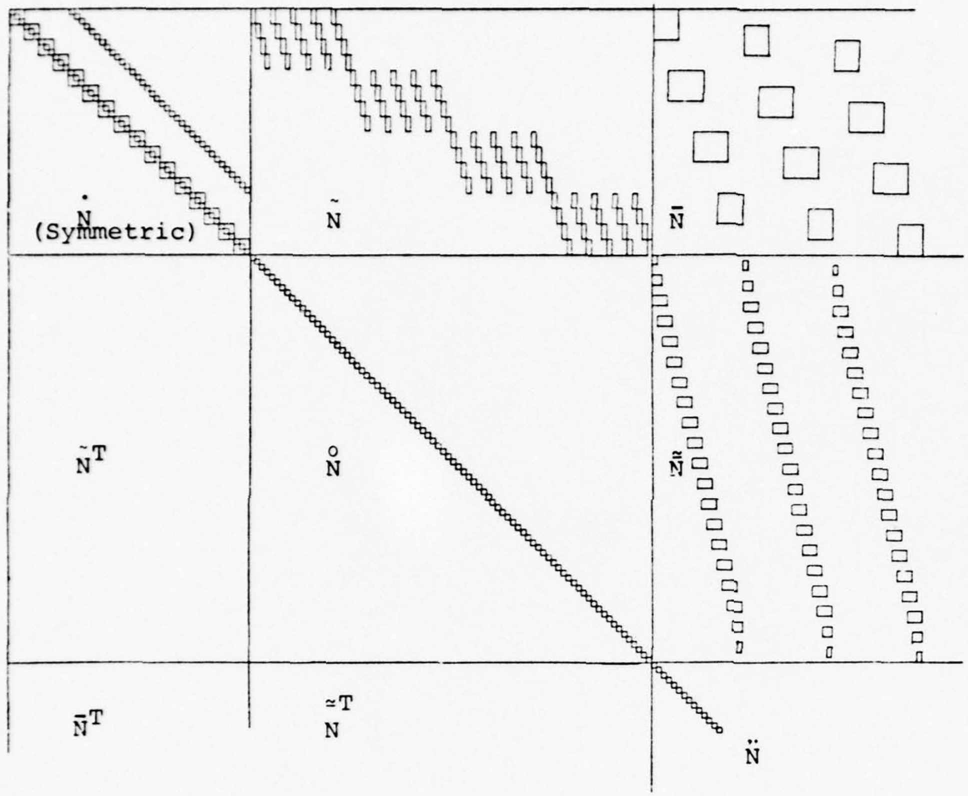

FIGURE 4.6.3-6a

Normal equations for Case II

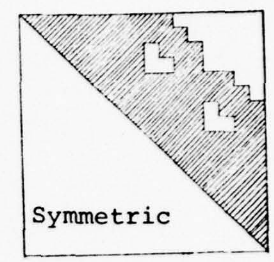

FIGURE 4.6.3-6b

Results of double fold operation for Case II second folding are illustrated in Figure 4.6.3-5. Note that in this figure the matrix is also a banded structure with a very large bandwidth compared to the original (Figure 4.6.3-4). However recursive partitioning can be employed again to solve this system of equations.

Figure 4.6.3-6a represents the normal equations resulting from the Case II example, namely, the interchange of the $\dot{N}$ and $\overset{o}{N}$ portions of the normal equations. The same double folding algorithm is again employed to solve this system of equations. The results of this operation are illustrated in Figure 4.6.3-3b.

There are many additional ways to arrange the data within the $\dot{N}$ and $\overset{o}{N}$ matrices which effect the structure of the first folding and second folding. Much more analysis is needed to determine the utility of this procedure in practice.

4.6.4 Autoregressive Modeling

In Section 4.6.1 a detailed description of how various systematic error models could be directly introduced into the photogrammetric block adjustment formulation were presented. All of these models assumed that the basic observations were independent from one another or were functionally related. There is another class of observations which are occasionally available for the photogrammetric problem. These are observations which are serially correlated. The technique employed in SAGA, (Brown & Trotter, 1969) called Autoregressive Feedback for including serially correlated range observations can also be employed in the photogrammetric block adjustment.

This technique is particularly well suited for observations which are time dependent and occur at a very high frequency such as the GPS positional data discussed in Section 4.3.1 or the inertial survey data described in Section 4.4.2.

The following sections present the basic mathematical development of the autoregressive feedback technique with emphasis on the GPS type observations.

4.6.4.1 The Autoregressive Model

A stationary sequence of serially correlated errors $\varepsilon_1, \varepsilon_{1-1}, \varepsilon_{1-2}, \ldots$ is governed by an autoregressive process in which

$$\varepsilon_1 = \alpha_1 \varepsilon_{1-1} + \alpha_2 \varepsilon_{1-2} + \ldots + \alpha_p \varepsilon_{1-p} + \eta_1 \qquad 4.6.4\text{-}1$$

where α's are constant coefficients and η_1 is a random impulse of zero mean and variance σ^2. According to this process, the i th error in the sequence is generated as a fixed linear combination with a superimposed, strictly random impulse. The process indicated in equation (4.6.4-1) is said to be the of pth order because p coefficients are involved in its description. Specific examples of the character of errors generated by various first order autoregressive processes are to be found in Brown, Bush, Sibol (1963). DBA's past experience with autoregressive feedback on systems similar to GPS indicates that a low order process is probably sufficient to model the serial correlation in GPS data.

4.6.4.2 Inverse Covariance Matrix of Autoregressive Process

As pointed out in earlier work, the practical utility of the autoregressive process in the adjustment of observations stems from the fact that, for a given set of autoregressive coefficients (which, can be estimated from the observations), the inverse of the covariance matrix of the observations can be computed analytically. Equally important, the inverse is a multidiagonal matrix, in which the number of diagonals are equal to 2p+1 for a process of the p-th order. In Brown, Bush, Sibol (1964), it was demonstrated that the basic result of this effect (derived originally by Wise, 1955) could be put into a more convenient form. Namely, if Λ denotes the covariance matrix of an autoregressive process governed by coefficients $\alpha_1, \alpha_2, \ldots, \alpha_p$, then Λ^{-1} can be expressed as the following product of lower and upper triangulatar matrices;

$$\sigma^2 \Lambda^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ \alpha_1 & -1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & \alpha_1 & -1 & 0 & \cdots & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \alpha_{n-4} & \cdots & -1 \end{bmatrix} \begin{bmatrix} -1 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} \\ 0 & -1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} \\ 0 & 0 & -1 & \alpha_1 & \cdots & \alpha_{n-3} \\ 0 & 0 & 0 & -1 & \cdots & \alpha_{n-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \tag{4.6.4-2}$$

in which $\alpha_n=0$ for $n>p$. If this result is applied, for instance, to a third order process involving coefficients $\alpha_1, \alpha_2, \alpha_3$, one finds, upon performing the above matrix multiplication that Λ^{-1} assumes the form

$$\sigma^2 \Lambda^{-1} = \begin{bmatrix} a_{-3} & b_{-2} & c_{-1} & d & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{-2} & a_{-2} & b_{-1} & c & d & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_{-1} & b_{-1} & a_{-1} & b & c & d & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ d & c & b & a & b & c & d & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & d & c & b & a & b & c & d & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & b & a & b & c & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & b & a & b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & b & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a & b & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b & a_{-1} & b_{-1} & c_{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & b_{-1} & a_{-2} & b_{-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & c_{-1} & b_{-2} & a_{-3} \end{bmatrix} \tag{4.6.4-3}$$

in which;

$$
\begin{array}{llll}
a = 1 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & a_{-1} = 1 + \alpha_1^2 + \alpha_2^2 & a_{-2} = 1 + \alpha_1^2 & a_{-3} = 1 \\
b = -\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3 & b_{-1} = -\alpha_1 + \alpha_1\alpha_2 & b_{-2} = -\alpha_1 & \\
c = -\alpha_2 + \alpha_2\alpha_3 & c_{-1} = -\alpha_2 & & \\
d = -\alpha_3 & & &
\end{array}
\qquad 4.6.4\text{-}4.
$$

This illustration is sufficiently general to demonstrate that certain properties of the inverse are of pivotal importance to the concept of autoregressive feedbacks. The properties are:

(1) the inverse is multi-diagonal, the number of diagonals being equal to 7 for p=3 (or equal to 2p+1 in general).

(2) except for pxp submatrices comprising the upper and lower corners of the matrix, the elements of each diagonal are constant, the number of different constants being equal to p, the order of the process.

(3) the formulas for the elements of the matrix are subject to a systematic, easily generalized development.

Expressions for second and first order processes can be obtained from the above development by sucessively equating α_3 and α_2 equal to zero in equation 4.6.4-4

4.6.4.3 Estimation of Autoregressive Coefficients

Let us assume that, by some means one has obtained a vector of residuals $(v_1, v_2, \ldots, v_n)$ that constitutes an estimate of the vector of actual errors $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$. One can then generate a set of autocorrelation coefficients $\rho_1, \rho_2 \ldots$ in the usual manner from;

$$\rho_k = \frac{\Sigma\, v_i\ v_{i-k}}{\Sigma\, v_i^2} \qquad 4.6.4\text{-}5$$

The autoregressive function can now be written in terms of residuals;

$$v_i = \alpha_1 v_{i-1} + \alpha_2 v_{i-2} + \ldots + \alpha_p v_{ip} + \eta_i \qquad 4.6.4\text{-}6$$

and this can be regarded as defining a set of observational equations involving the autoregressive coefficients as parameters. The least squares adjustment technique can be employed to generate a system of normal equations for the determination of the α's. These turn out to assume the form;

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_p \\ \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-1} \\ \rho_2 & \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-2} \\ \rho_3 & \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \rho_p & \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \vdots \\ \rho_p \end{bmatrix} \qquad 4.6.4\text{-}7$$

In practice, it is desirable to establish the autoregressive process of lowest order which satisfactorily models the observed process. This can be accomplished by beginning with a first order process which, from equation 4.6.4-7 would lead to the estimate;

$$\alpha_1 = \rho_1 . \qquad 4.6.4\text{-}8$$

Using this estimate of α_i the secondary residual η_i can be computed as:

$$\eta_i = \epsilon_i - \alpha_i \epsilon_{i-1} \quad . \qquad 4.6.4\text{-}9$$

If these secondary residuals turn out to be serially uncorrelated, the first order process can be accepted as satisfactory. To determine whether or not the η's are independent, their first order coefficient of correlation will be computed from;

$$r = \frac{\Sigma \eta_i \eta_{i-1}}{\Sigma \eta_i^2} \quad . \qquad 4.6.4\text{-}10$$

The statistical signifance of r can be tested using the well known result that the quantity;

$$z = \tfrac{1}{2} \ln \frac{1+r}{1-r} \qquad 4.6.4\text{-}11$$

is approximately normally distributed with mean and variance of

$$\mu_z = \tfrac{1}{2} \ln \frac{1+\rho}{1-\rho} \qquad 4.6.4\text{-}12$$

$$\sigma_z^2 = 1/(n-3) \qquad 4.6.4\text{-}13$$

in which ρ is the true, but unknown correlation coefficient. It follows that, at the 95% level of confidence, r will differ significantly from zero only if

$$z > 1.96/\sqrt{n-3} \quad . \qquad 4.6.4\text{-}14$$

If r should turn out to be significantly different from zero, then an attempt to model the system using a second order process would be initiated. Here, according to equation 4.6.4-7, the coefficients would be established from

$$\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \qquad 4.6.4\text{-}15$$

which has the solution

$$\begin{aligned} \alpha_1 &= (\rho_1 - \rho_1 \rho_2)/(1-\rho_1^2), \\ \alpha_2 &= (-\rho_1^2 + \rho_2)/(1-\rho_1^2). \end{aligned} \qquad 4.6.4\text{-}16$$

The secondary residuals for the second order process are

$$\eta_i \doteq \epsilon_i - (\alpha_1 \epsilon_{i-1} + \alpha_2 \epsilon_{i-2}). \qquad 4.6.4\text{-}17$$

The serial independence of these residuals would be tested in precisely the same manner as described above for the first order process.

The above procedure would be repeated until a satisfactory fit is obtained. Once a satisfactory autoregressive model has been established, the spectral density function can be computed analytically from the following elegant result derived from Wise:

$$\nu(\theta) = \sigma^2/(-1 + \alpha_1 e^{i\theta} + \alpha_2 e^{i2\theta} + \ldots + \alpha_p e^{ip\theta})(-1 + \alpha_1 e^{-i\theta} + \alpha_2 e^{-i2\theta} + \ldots + \alpha_p e^{-ip\theta})$$

4.6.4-18

where $i = \sqrt{-1}$ and θ assumes the discrete sample values,

$$\theta = \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \ldots, \frac{2(n-1)}{n}\pi, 2\pi . \qquad 4.6.4\text{-}19$$

If Δt denotes the time between successive points and $T = n\Delta t$ denotes the total time span, θ may be put in the form,

$$\theta_j = \frac{2j\Delta t}{T}\pi, \quad j = 1, 2, \ldots, n \qquad 4.6.4\text{-}20$$

where θ_j may be said to correspond to the frequency of $1/2_j\Delta t$ cycles per second. For the first order process, equation 4.6.4-18 reduces to the following well known expression for the spectral density function of the damped exponential autocorrelation function.

$$\nu(\theta) = \sigma^2/(1 - 2\rho\cos\theta + \rho^2) \qquad 4.6.4\text{-}21$$

4.6.4.4 Refined Normal Equations

From the foregoing, it can be seen that a successful autoregressive analysis of residuals provides the solution to two central problems of random error analysis: (1) the determination of the inverse covariance matrix of the observational vector and (2) the determination of the spectral density function of the error process. It remains to be shown precisely how this information can be used in a refinement of the adjustment and what its use entails in the way of additional complication. For this purpose the specific case of an adjustment of a single channel of observations governed by a first order autoregressive process shall be considered. This is sufficient to demonstrate the general principles of the operation. Accordingly,

$$\rho\epsilon_{i-1} + \eta_i \qquad 4.6.4\text{-}22$$

define a first order process in which;

$$E(\eta_i) = 0 \qquad 4.6.4\text{-}23$$

$$\text{var}(\eta_i) = \sigma^2 \text{ (variance of high frequency component of noise).} \qquad 4.6.4\text{-}24$$

$$\text{cov}(\eta_i, \eta_{i-k}) = 0 \quad \text{for all } k > 0 \qquad 4.6.4\text{-}25$$

Then it is easily shown that;

$$E(\epsilon_i) = 0 \, , \qquad 4.6.4\text{-}26$$

$$\text{var}(\epsilon_i) = \sigma^2/(1-\rho^2) = \text{total error variance,} \qquad 4.6.4\text{-}27$$

$$\text{cov}(\epsilon_i, \epsilon_{i-k}) = \frac{\rho^k \sigma^2}{1-\rho^2} \, , \qquad 4.6.4\text{-}28$$

$$\text{cor}(\epsilon_i, \epsilon_{i-k}) = \rho^k \, . \qquad 4.6.4\text{-}29$$

It follows that the covariance matrix Λ of an n vector of errors is;

$$\Lambda = \frac{\sigma^2}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^n \\ \rho & 1 & \rho & \cdots & \rho^{n-1} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho^n & \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{bmatrix} \qquad 4.6.4\text{-}30$$

Now assume that the observational vector is employed in an adjustment. The normal equations can then be expressed as;

$$(B^T \Lambda^{-1} B)\delta = B^T \Lambda^{-1} \epsilon \qquad 4.6.4\text{-}31$$

in which δ denotes the vector of parametric corrections and the matrix B and the vector ε can be decomposed into

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \qquad 4.6.4\text{-}32$$

where the b's are row vectors of order equal to that of δ and the ε's are scalars. In a conventional adjustment the covariance matrix is taken to be diagonal (ρ=0), and its inversion is therefore trivial. In the present instance, however, the covariance matrix is completely filled (eq. 4.6.4-30), a fact which introduces complications. Because Λ in this instance is generated by a first order autoregressive process, the results of 4.6.4.3 may be employed to immediately write the following expression for Λ^{-1}.

$$\Lambda^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 1 & -\rho & 0 & 0 & \cdots & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \cdots & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & \cdots & 0 \\ 0 & 0 & -\rho & 1+\rho^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad 4.6.4\text{-}33$$

The consequences of the patterned and regular structure of Λ^{-1} shall now be traced through the least squares reduction. First, note that Λ^{-1} can be decomposed as follows ;

$$\sigma^2\Lambda^{-1} = (1+\rho^2)I - \rho U - \rho U^T - \rho^2\Delta \qquad 4.6.4\text{-}34$$

In which I is an nxn unit matrix and U and Δ are nxn matrices of the form ;

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \qquad 4.6.4\text{-}35$$

N is now defined as the coefficient matrix of the normal equations,

$$N = B^T\Lambda^{-1}B$$

$$= \frac{1}{\sigma^2}(b_1^T\; b_2^T\; \dots\; b_n^T)\left[(1+\rho^2)\,I - \rho U - \rho U^T - \rho^2\Delta\right]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad 4.6.4\text{-}36$$

This expands to

$$\begin{aligned} N = \frac{1}{\sigma^2}\Big[&(1+\rho^2)(b_1^T b_1 + b_2^T b_2 + \dots + b_n^T b_n) \\ &-\rho\,(b_1^T b_2 + b_2^T b_3 + \dots + b_{n-1}^T b_n) \\ &-\rho\,(b_2^T b_1 + b_3^T b_2 + \dots + b_n^T b_{n-1}) \\ &-\rho^2\,(b_1^T b_1 + b_n^T b_n)\Big] \end{aligned} \qquad 4.6.4\text{-}37$$

If we define ,

$$\Delta b_i = b_{i+1} - b_i \qquad 4.6.4\text{-}38$$

and note that then ,

$$b_{i-1}^T b_i = b_{i-1}^T (b_{i-1} + \Delta b_{i-1}) = b_{i-1}^T b_{i-1} + b_{i-1}^T \Delta b_{i-1} \, , \qquad 4.6.4\text{-}39$$

$$b_i^T b_{i-1} = b_i^T (b_i - \Delta b_{i-1}) = b_i^T b_i - b_i^T \Delta b_{i-1} \, , \qquad 4.6.4\text{-}40$$

it follows that ,

$$\begin{aligned} b_{i-1}^T b_i + b_i^T b_{i-1} &= b_{i-1}^T b_{i-1} + b_i^T b_i + (b_{i-1}^T - b_i^T) \Delta b_{i-1} \\ &= b_{i-1}^T b_{i-1} + b_i^T b_i + \Delta b_{i-1}^T \Delta b_{i-1} \, . \end{aligned} \qquad 4.6.4\text{-}41$$

When this result is substituted into 4.6.4-37 and appropriate algebraic manipulations are performed, one obtains

$$\begin{aligned} N = \frac{1}{\sigma^2} \Big[& (1 - 2\rho + \rho^2)(b_1^T b_1 + b_2^T b_2 + \ldots + b_n^T b_n) \\ & + \rho \;\; (\Delta b_1^T \Delta b_1 + \Delta b_2^T \Delta b_2 + \ldots + \Delta b_{n-1}^T \Delta b_{n-1}) \\ & + \rho \;\; (1-\rho)(b_1^T b_1 + b_n^T b_n) \Big] \end{aligned} \qquad 4.6.4\text{-}42$$

which is the same as

$$N = \frac{1}{\sigma^2} \left[(1-\rho)^2 \, B^T B + \rho \, \Delta B^T \Delta B + \rho(1-\rho)(b_1^T b_1 + b_n^T b_n) \right] \qquad 4.6.4\text{-}43$$

in which ;

$$\Delta B = \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \vdots \\ \Delta b_{n-1} \end{bmatrix} = \begin{bmatrix} b_2 - b_1 \\ b_3 - b_2 \\ \vdots \\ b_n - b_{n-1} \end{bmatrix} \qquad 4.6.4\text{-}44$$

In an analogous manner one obtains for the right side of the normal equations

$$c = B^T \Lambda^{-1} \epsilon$$

$$= \frac{1}{\sigma^2} \left[(1-\rho^2)\, B^T \epsilon + \rho\, \Delta B^T \Delta\epsilon + \rho(1-\rho)(b_1^T \epsilon_1 + b_n^T \epsilon_n) \right] \qquad 4.6.4\text{-}45$$

in which;

$$\Delta\epsilon = \begin{bmatrix} \Delta\epsilon_1 \\ \Delta\epsilon_2 \\ \\ \Delta\epsilon_{n-1} \end{bmatrix} = \begin{bmatrix} \epsilon_2 - \epsilon_1 \\ \epsilon_3 - \epsilon_2 \\ \\ \epsilon_n - \epsilon_{n-1} \end{bmatrix} \qquad 4.6.4\text{-}46$$

Equations 4.6.4-43 and 4.6.4-45 are the results sought. In analyzing them, it is first noted that when $\rho=0$, the normal equations reduce to the usual expression:

$$\frac{1}{\sigma^2} (B^T B)\, \delta = \frac{1}{\sigma^2} B^T \epsilon \ . \qquad 4.6.4\text{-}47$$

Next , note that since ΔB and $\Delta\epsilon$ are of first order, the expressions $\Delta B^T \Delta B$ and $\Delta B^T \Delta\epsilon$ are of second order and one can assert that ;

$$B^T B \gg \Delta B^T \Delta B, \qquad 4.6.4\text{-}48$$

$$B^T \epsilon \gg \Delta B^T \Delta \epsilon$$

This implies that as long as ρ is not too close to unity, the normal equations will be dominated by the leading terms and the solution vector δ will be only slightly dependent on the value of ρ. Hence, in some situations the solution is but weakly affected by the presence of even rather moderate serial correlation. Be this as it may, the covariance matrix of the solution vector is very strongly dependent on the degree of serial correlation. Then, when $B^T B$ dominates the normal equations, the covariance matrix of the parametric vector is given by

$$\Sigma \simeq \frac{\sigma^2}{(1-\rho)^2} (B^T B)^{-1} . \qquad 4.6.4\text{-}49$$

Thus if ρ were actually equal to 0.9 and one were to ignore this fact in the adjustment, the solution vector itself would probably not be very much affected, for the factor $(1-\rho)^2$ of the leading terms on both sides of the normal equations would cancel out , but the covariance matrix associated with the solution would be incorrect by a factor 100 (or $1/(1-\rho)^2$).

Because of the relative insensitivity of the solution vector to serial correlation, the residuals from the adjustment are also relatively insensitive to serial correlation (remember an adjustment restricted to a single observational vector r is considered herein). This means that it is a sound and defensible practice to estimate the autoregressive function from residuals of a preliminary adjustment in which the observational errors are initially considered to be uncorrelated. The autoregressive function thus initially determined can then be fedback into the solution according to the development of this section and the resulting, more nearly correct normal equations can be solved to generate fresh residuals for a more refined autoregressive analysis. Clearly, this process can be iterated to stability (ordinarily, a single iteration is sufficient). Thus, autoregressive feedback is an adaptive process that ultimately becomes independent of initial or apriori estimates of noise variance.

Returning to the normal equations 4.6.4-43, 4.6.4-45, it is noted that the extra computations entailed by rigorous consideration of a first order autoregressive process are surprisingly minor. The B^TB and $B^T\varepsilon$ terms, being of similar form, follow the same logic. Moreover, as is clear from 4.6.4-42, the equations can be formed in a cumulative manner: the i-1 st step of which would involve only observational equations from the i-th and 1+1 st observations. Hence, the formation of the normal equations for a first order process entails a scheme in which a moving pair of successive observational equations are processed at each step; similarily, a second order process entails a scheme in which a moving triplet of successive observational equations are processed at each step, and so on. A major benefit to be derived from the admission of an autoregressive process into the adjustment is the attainment of more realistic results from error propagation. In particular, it would permit one to extract whatever advantages are to be gained from moderately high sampling rates without paying the usual penalty of an absurdly optimistic estimation of output accuracies.

The process outlined here is very general in nature and can be applied to many types of high frequency observations. DBA has successfully implemented this algorithm in our short arc tracking reduction program, SAGA, for handling serially correlated ranging observations. The techniques could be easily adapted to the GPS data which, if collected at a high frequency, could exhibit high serial correlation.

4.6.5 Automatic Editing

As pointed out in Section 2.7 and 3.7 Automatic Editing features have been included in mnay photogrammetric triangulation programs. However, all require that a new formation and solution be performed each time one or more points is either edited or reinserted.

A technique was developed by Mikhail and Helmering in 1973, whereby points could be edited and only their contribution subtracted or added into a set of previously formed normal equations. Helmering's technique has been experimentally tested on the TA-3/PA at DMAAC and was demonstrated to save a tremendous amount of time in the real time editing of data. However, the technique has never, to our knowledge, been applied to a photogrammetric block adjustment. Helmering's Doctorial dissertation adequately develops the mathematics for the three photo algorithm. This development is easily expanded to the more general case and will not be presented herein.

Although not an automatic editing scheme, another technique which is not currently utilized in photogrammetric triangulation is, interactive editing via a CRT display screen. The basic concept is to first store the results from a triangulation on either a magnetic tape or disc file. This tape would then be input to an editing program,whereby the observational residuals could be plotted in a variety of formats which would be operator selectable. The operator could then edit a contribution of this observation from the stored normal equation and constraint column and resolve the system. The results would then be displayed for further editing.

Another automatic editing technique is possible for image measurement which is based on the pair-wise intersection of multiple ray points. DBA has used this technique for manual data editing very successfully. The technique shows tremendous potential in automatically detecting image measurement errors.

The combination of the three approaches outlined above would result in a procedure which would definitely increase the overall throughput of a triangulation process.

5.0 CONCLUSIONS AND RECOMMENDATIONS

In the introduction to this report it was stated that the overall objective of this report is an identification of new technology which could either increase the targeting accuracy capabilities for the MC&G Community or to increase the overall Photogrammetric Triangulation throughput. A number of specific shortfalls in the existing triangulation process have been identified, many of which are beyond the scope of this study to analyze in detail.

The primary shortfall identified herein is the inability of current techniques to exploit all of the information available for triangulation. This includes auxiliary sensor data and photographic imagery data. In general, this information can be included into an adjustment in a number of methods; direct observations, functional representation, error modeling or statistically. The general procedure for including each of these types of models are included herein with specific examples of what types of information could be included with each model. Many of these procedures result in a set of normal equations which typically exhibit a banded-bordered structure. Although solutions for this type of normal equation system have been in existence for many years, no attempt has been made to develop an efficient algorithm which exploits the specific structure of the border itself. An algorithm for this is developed in Section 4.6.3 of this report.

Since the objective of this report is to identify technologies for future use, no attempt has been made to quantify the exact benefits which one could expect to derive by implementing the techniques. Instead, a set of alternatives have been identified, many of which need much more analysis and testing before their specific benefit can be realistically assessed. Listed below are areas which DBA believes worthy of future indepth study.

1. Simulation testing of the various techniques for incorporating GPS positioning data.

   a. Error Modeling

   b. Spline type functional modeling

   c. Autoregressive feedback modeling

2. Implementation and testing of the banded-bordered solution algorithm.

3. Automatic reordering of the normal equations with emphasis on handling the border.

4. Geometric constraints by the method of continuous traces.

5. Use of Mini-computers and/or interactive systems for:

   a. Material availability
   b. Coverage Planning and
   c. Automatic editing.

Other techniques which are identified in this report can be implemented without further study. These include:

1. Distance, Azimuth and elevation constraints, and

2. Camera Calibration and film deformation models.

A single study could possibly cover the first three items. In that during the GPS spline modeling technique phase a banded-bordered set of normal equations can be formulated as in Section 4.6.3. This system could then be solved by use of standard methods as well as the double fold technique. The results could then be compared with respect to accuracy and run times. If the double fold technique is practical, automatic reordering algorithms can be investigated to minimize the run time by rearranging the N and $\overset{\circ}{N}$ elements to maximize the efficiency of the algorithm.

Another simulation type study should be undertaken to develop and test the algorithm for geometric constraint type modeling using continuous traces. This study could include a small set of three photographs of real data to validate the approach.

A third area relates directly to the overall triangulation process throughput. This is the use of interactive mini-computer stations for analyzing the Material Available, Coverage Planning and Editing aspects. As pointed out in Sections 2 and 3, these steps are currently performed manually and are very tedious and time consuming. With the current technology and the low cost of small computers, there exists the potential to greatly reduce the tedium and time through automation.

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METRIC SYSTEM

BASE UNITS:

Quantity	Unit	SI Symbol	Formula
length	metre	m	...
mass	kilogram	kg	...
time	second	s	...
electric current	ampere	A	...
thermodynamic temperature	kelvin	K	...
amount of substance	mole	mol	...
luminous intensity	candela	cd	...

SUPPLEMENTARY UNITS:

Quantity	Unit	SI Symbol	Formula
plane angle	radian	rad	...
solid angle	steradian	sr	...

DERIVED UNITS:

Quantity	Unit	SI Symbol	Formula
Acceleration	metre per second squared	...	m/s
activity (of a radioactive source)	disintegration per second	...	(disintegration)/s
angular acceleration	radian per second squared	...	rad/s
angular velocity	radian per second	...	rad/s
area	square metre	...	m
density	kilogram per cubic metre	...	kg/m
electric capacitance	farad	F	A·s/V
electrical conductance	*siemens*	S	A/V
electric field strength	volt per metre	...	V/m
electric inductance	henry	H	V·s/A
electric potential difference	volt	V	W/A
electric resistance	ohm		V/A
electromotive force	volt	V	W/A
energy	joule	J	N·m
entropy	joule per kelvin	...	J/K
force	newton	N	kg·m/s
frequency	hertz	Hz	(cycle)/s
illuminance	lux	lx	lm/m
luminance	candela per square metre	...	cd/m
luminous flux	lumen	lm	cd·sr
magnetic field strength	ampere per metre	...	A/m
magnetic flux	weber	Wb	V·s
magnetic flux density	tesla	T	Wb/m
magnetomotive force	ampere	A	...
power	watt	W	J/s
pressure	pascal	Pa	N/m
quantity of electricity	coulomb	C	A·s
quantity of heat	joule	J	N·m
radiant intensity	watt per steradian	...	W/sr
specific heat	joule per kilogram-kelvin	...	J/kg·K
stress	pascal	Pa	N/m
thermal conductivity	watt per metre-kelvin	...	W/m·K
velocity	metre per second	...	m/s
viscosity, dynamic	pascal-second	...	Pa·s
viscosity, kinematic	square metre per second	...	m/s
voltage	volt	V	W/A
volume	cubic metre	...	m
wavenumber	reciprocal metre	...	(wave)/m
work	joule	J	N·m

SI PREFIXES:

Multiplication Factors	Prefix	SI Symbol
1 000 000 000 000 = 10^{12}	tera	T
1 000 000 000 = 10^{9}	giga	G
1 000 000 = 10^{6}	mega	M
1 000 = 10^{3}	kilo	k
100 = 10^{2}	hecto*	h
10 = 10^{1}	deka*	da
0.1 = 10^{-1}	deci*	d
0.01 = 10^{-2}	centi*	c
0.001 = 10^{-3}	milli	m
0.000 001 = 10^{-6}	micro	μ
0.000 000 001 = 10^{-9}	nano	n
0.000 000 000 001 = 10^{-12}	pico	p
0.000 000 000 000 001 = 10^{-15}	femto	f
0.000 000 000 000 000 001 = 10^{-18}	atto	a

* To be avoided where possible.

MISSION
of
Rome Air Development Center

RADC plans and conducts research, exploratory and advanced development programs in command, control, and communications (C^3) activities, and in the C^3 areas of information sciences and intelligence. The principal technical mission areas are communications, electromagnetic guidance and control, surveillance of ground and aerospace objects, intelligence data collection and handling, information system technology, ionospheric propagation, solid state sciences, microwave physics and electronic reliability, maintainability and compatibility.

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